A SHORT COURSE

IN

INTERPOLATION

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BY

E. T. WHITTAKER
Sc.D. ; F.R.S.
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF EDINBURGH

AND

GEORGE ROBINSON
M.A., B.Sc.
LECTURER IN MATHEMATICS IN THE UNIVERSITY OF EDINBURGH

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PREFACE

A KNOWLEDGE of the Theory of Interpolation is required by all who make inferences from the results of observation, especially by astronomers, physicists, statisticians, and actuaries. Until recently it was somewhat neglected in the mathematical schools of many British Universities; but of late years there has been wider acceptance of the view that the subject is easy enough to be put at the beginning of a student's course, that it forms an excellent preparation for the Differential Calculus, and that it cannot be left out.

The present text is offered as a short exposition suitable for first-year undergraduates: it is a separate issue of the first four chapters of a larger work by the same authors, dealing with the general field of the Calculus of Observations.

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CHAPTER I

INTERPOLATION WITH EQUAL INTERVALS OF THE ARGUMENT

1. **Introduction.**—Mathematics is occupied largely with the idea of *correspondence*: *e.g.* to every number x there corresponds a value of x^2, thus

$$x = 1, 2, 3, 4, 5, \ldots$$
$$x^2 = 1, 4, 9, 16, 25, \ldots$$

One of the two variables between which correspondence holds is called the *argument* and the other is called the *function* of that argument.

If a function y of an argument x is defined by an equation $y = f(x)$, where $f(x)$ is an algebraical expression involving only arithmetical operations such as squaring, dividing, etc., then by performing these operations we can find accurately the value of y, which corresponds to any value of x. But if $y = \log_{10} x$ (say), it is not possible to calculate y by performing simple arithmetical operations on x (at any rate it is not possible to calculate y accurately by performing a finite number of such operations), and we are compelled to have recourse to a *table*, which gives the values of y corresponding to certain selected values of x; *e.g.*

x.	log x.	x.	log x.
7·0	0·845 098	7·4	0·869 232
7·1	0·851 258	7·5	0·875 061
7·2	0·857 332	7·6	0·880 814
7·3	0·863 323	7·7	0·886 491

The question then arises as to how we can find the values of the function log x for values of the argument x which are intermediate between the tabulated values, *e.g.* such a value as $x=7{\cdot}152$. The answer to this question is furnished by the theory of *Interpolation*, which in its most elementary aspect may be described as the science of "reading between the lines of a mathematical table."

In the further development of the theory of interpolation it will be shown how to find the differential coefficient of a function which is specified by a table, and also to find its integral taken between any bounds of integration.

A kind of interpolation was used by Briggs,* but interpolation of the kind hereafter explained, based on the representation of functions by polynomials, was first introduced by James Gregory † in 1670.

2. **Difference Tables.**—Suppose a function $f(u)$ is given in a table for the values a, $a+w$, $a+2w$, $a+3w$, . . . of its argument u. It is required to find the value of the function when the argument has the value $a+xw$, where x is a fraction.

Before this problem can be solved by the method of interpolation, it is first necessary to form what are called the *differences* of the tabular values. The quantity

$$f(a+w)-f(a)$$

is denoted by $\Delta f(a)$ and is called the *first difference* of $f(a)$. The first difference of $f(a+w)$ is $f(a+2w)-f(a+w)$, which is denoted by $\Delta f(a+w)$. Moreover, the quantity

$$\Delta f(a+w)-\Delta f(a)$$

is denoted by $\Delta^2 f(a)$ and is called the *second difference* of $f(a)$, while the quantity

$$\Delta^2 f(a+w)-\Delta^2 f(a)$$

is denoted by $\Delta^3 f(a)$ and is called the *third difference* of $f(a)$, and so on.

It is convenient to arrange the tabular values and their differences for increasing values of the argument in what is called a *difference* table, as follows:

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* Briggs' method was, however, closely related to the modern central-difference formulae. Cf. his *Arithmetica Logarithmica*, ch. xiii., and his *Trigonometria Britannica*, ch. xii. Cf. *Journal of the Institute of Actuaries*, **14**, pp. 1, 73, 84, 88; **15**, p. 312.

† Rigaud's *Correspondence of Scientific Men of the 17th Century*, **2**, p. 209.

Argument.	*Entry.*	Δ.	Δ^2.	Δ^3.
a	$f(a)$			
		$\Delta f(a)$		
$a+w$	$f(a+w)$		$\Delta^2 f(a)$	
		$\Delta f(a+w)$		$\Delta^3 f(a)$
$a+2w$	$f(a+2w)$		$\Delta^2 f(a+w)$	
		$\Delta f(a+2w)$		$\Delta^3 f(a+w)$
$a+3w$	$f(a+3w)$		$\Delta^2 f(a+2w)$	
		$\Delta f(a+3w)$		$\Delta^3 f(a+2w)$
$a+4w$	$f(a+4w)$		$\Delta^2 f(a+3w)$	

and similarly for differences of order higher than the third. The first entry $f(a)$ is called the *leading term*, and the differences of $f(a)$, that is to say $\Delta f(a)$, $\Delta^2 f(a)$, . . . are called the *leading differences.* Evidently *each difference in the table is the number* (*with its proper algebraic sign*) *obtained by subtracting the number immediately above and to the left from the number immediately below and to the left.*

The sum of the entries in any column of differences is equal to the difference between the first and last entries of the preceding column. This affords a numerical check on the accuracy of the table. Thus in the above table we have

$$\Delta^2 f(a+3w) = \Delta^2 f(a) + \Delta^3 f(a) + \Delta^3 f(a+w) + \Delta^3 f(a+2w).$$

An example of a difference table is the following, which represents the natural sines of angles from 25° 40′ 0″ to 25° 43′ 0″ inclusive at intervals of 20″.

Argument.	*Entry.*	Δ.	Δ^2.	Δ^3.
25° 40′ 0″	0·43313 47858 66963			
		8 73933 05476		
20″	0·43322 21791 72439		– 40 73056	
		8 73892 32420		– 822
40″	0·43330 95684 04859		– 40 73878	
		8 73851 58542		– 822
25° 41′ 0″	0·43339 69535 63401		– 40 74700	
		8 73810 83842		– 820
20″	0·43348 43346 47243		– 40 75520	
		8 73770 08322		– 823
40″	0·43357 17116 55565		– 40 76343	
		8 73729 31979		– 821
25° 42′ 0″	0·43365 90845 87544		– 40 77164	
		8 73688 54815		– 821
20″	0·43374 64534 42359		– 40 77985	
		8 73647 76830		– 822
40″	0·43383 38182 19189		– 40 78807	
		8 73606 98023		
25° 43′ 0″	0·43392 11789 17212			

It will be seen that in this case the third differences are practically constant when quantities beyond the fifteenth place are neglected, any departure from constancy in the last place being really due to the neglect of the sixteenth place of decimals in the original entries. So the fourth differences are zero.

It will be found that *in the case of practically all tabular functions the differences of a certain order are all zero; or, to speak more accurately, they are smaller than one unit in the last decimal place retained in the tables in question.* This fact lies at the basis of the method of interpolation, as we shall now see.

3. **Symbolic Operators.**—The formulae of the calculus of differences may be very simply represented by the use of what are called *symbolic operators.* Of these we have already introduced Δ, and we shall now consider another operator denoted by E.

Let w represent the interval between successive values of the argument of the function $f(a)$, and let E denote the operation of increasing the argument by w, so that $\mathrm{E}f(a)=f(a+w)$; in general we shall write $\mathrm{E}^x f(a)=f(a+xw)$, where x is an integer. Now by definition we had $\Delta f(a+xw)=f(a+xw+w)-f(a+xw)$, so $\Delta f(a+xw)=(\mathrm{E}-1)f(a+xw)$. It is therefore evident that the operators E and Δ are connected by the relation $\Delta=\mathrm{E}-1$ or

$$\mathrm{E}=1+\Delta.$$

When symbolic operators obey the ordinary laws of Algebra they may be separated from the symbols representing the functions to which they refer and treated independently in much the same way as symbols of quantity. Now it may be easily shown that the following relations are true for the operator Δ:

$$\Delta\{f(a)+f(b)+f(c)+\ldots\}=\Delta f(a)+\Delta f(b)+\Delta f(c)+\ldots,$$

$$\Delta kf(a)=k\Delta f(a), \text{ where } k \text{ is a constant factor,}$$

$$\Delta^m\Delta^n f(a)=\Delta^{m+n}f(a), \text{ where } m, n \text{ are positive integers.}$$

The corresponding identities for E are:

$$\mathrm{E}\{f(a)+f(b)+f(c)+\ldots\}=\mathrm{E}f(a)+\mathrm{E}f(b)+\mathrm{E}f(c)+\ldots,$$

$$\mathrm{E}kf(a)=k\mathrm{E}f(a),$$

$$\mathrm{E}^m\mathrm{E}^n f(a)=\mathrm{E}^{m+n}f(a).$$

Thus in many respects the operators E and Δ behave like algebraic symbols and may be combined like them.

The following examples illustrate the use of these operators:

Ex. 1.—*To express the nth differences of a tabulated function in terms of the successive entries.*

$$\Delta^n f(a) = (E-1)^n f(a)$$
$$= \{E^n - nE^{n-1} + \frac{n(n-1)}{2!}E^{n-2} - \ldots + (-1)^n\}f(a),$$

i.e.

$$\Delta^n f(a) = f(a+nw) - nf(a+nw-w) + \frac{n(n-1)}{2!}f(a+nw-2w) - \ldots + (-1)^n f(a).$$

Ex. 2.—*To express the function $f(a+xw)$ in terms of $f(a)$ and the successive differences of $f(a)$, when x is a positive integer.*

$$f(a+xw) = E^x f(a)$$
$$= (1+\Delta)^x f(a),$$

so that

$$f(a+xw) = f(a) + x\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \ldots + \Delta^x f(a).$$

4. The Differences of a Polynomial.—We find without difficulty that the difference table for the function $y = x^3$ is as follows:

x.	y.	Δ.	Δ^2.	Δ^3.	Δ^4.
0	0				
		1			
1	1		6		
		7		6	
2	8		12		0
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

It will be seen that the third differences of this function are rigorously constant and the fourth differences are zero. This is a particular case of a general property which we shall now establish.

Note that the table may be extended indefinitely when we know the third differences to be constant. For by definition, when we add to an entry in a column of differences the corresponding first difference, the sum so formed gives the next entry in the column. It follows that the column of second differences can be formed from the leading term 6 by repeatedly adding the constant third difference 6; the column of first differences being formed from the leading term 1 by adding in succession the second differences 6, 12, 18, . . . The values of x^3 are then obtained from the leading term 0 by adding in succession the first differences 1, 7, 19, 37, 61, . . .

Consider the case when the tabulated function $f(x)$ is a polynomial of degree n, say,

$$f(x)=\mathrm{A}x^n+\mathrm{B}x^{n-1}+\mathrm{C}x^{n-2}+\ldots+\mathrm{L}x+\mathrm{M}.$$

Then

$$\begin{aligned}\Delta f(a)&=f(a+w)-f(a)\\&=\mathrm{A}\{(a+w)^n-a^n\}+\mathrm{B}\{(a+w)^{n-1}-a^{n-1}\}+\ldots+\mathrm{L}w.\end{aligned}$$

Now

$$(a+w)^n=a^n+nwa^{n-1}+\frac{n(n-1)}{2!}w^2a^{n-2}+\ldots+w^n,$$

so that

$$\begin{aligned}\Delta f(a)=\mathrm{A}\{&nwa^{n-1}+\frac{n(n-1)}{2!}w^2a^{n-2}+\ldots+w^n\}\\&+\mathrm{B}\{(n-1)wa^{n-2}+\frac{(n-1)(n-2)}{2!}w^2a^{n-3}+\ldots+w^{n-1}\}\\&+\ldots\\&+\mathrm{L}w.\end{aligned}$$

This is a polynomial of degree $(n-1)$ in a, and therefore the first differences of a polynomial represent another polynomial of degree less by one unit.

By repeated application of this result we see that

the 2nd differences represent a polynomial of degree $n-2$,
" 3rd " " " " $n-3$,
. . . .
" nth " " " " 0,

i.e. the nth differences are constant. It follows, therefore, that *the $(n+1)$th differences of a polynomial of the nth degree are all zero.*

5. **The Differences of Zero.**—A table of values of any power of the natural numbers may be formed by simple addition when the leading term and the leading differences are known, in

precisely the same way as in forming the table of cubes (§ 4). The differences of the leading term 0^p, which are generally used in forming a table of x^p, are known as *the differences of zero.* They are of frequent occurrence in the calculus of differences.

In order to form a table of reference of the differences of zero we apply the result of § 3 (Ex. 1),

$$\Delta^n f(a) = f(a+nw) - nf(a+nw-w) + \tfrac{1}{2}n(n-1)f(a+nw-2w) - \ldots$$

and write

$$\Delta^n x^p = (x+n)^p - n\{x+(n-1)\}^p + \tfrac{1}{2}n(n-1)\{x+(n-2)\}^p - \ldots$$

If we now substitute in this equation particular values for x, p, and n, we obtain the equations

$$\Delta^n 0^p = n^p - n(n-1)^p + \tfrac{1}{2}n(n-1)(n-2)^p - \ldots \pm n.1^p \mp 0^p,$$
$$\Delta^{n-1} 1^{p-1} = n^{p-1} - (n-1)^p + \tfrac{1}{2}(n-1)(n-2)^p - \ldots \pm 1^{p-1},$$

and therefore $$\Delta^n 0^p = n\Delta^{n-1}1^{p-1}. \qquad (1)$$

From the relation $\Delta^{n-1}f(a+w) = \Delta^n f(a) + \Delta^{n-1}f(a)$ we see that $\Delta^{n-1}1^{p-1} = \Delta^n 0^{p-1} + \Delta^{n-1}0^{p-1}$, and equation (1) may be written

$$\Delta^n 0^p = n(\Delta^n 0^{p-1} + \Delta^{n-1}0^{p-1}). \qquad (2)$$

We now construct a table of values of $\Delta^n 0^p$ by the repeated application of this equation, remembering that $\Delta^0 0^1 = 0$, $\Delta^1 0^1 = 1$, and also that $\Delta^n 0^p = 0$ for $n > p$.

p.	$\Delta 0^p$.	$\Delta^2 0^p$.	$\Delta^3 0^p$.	$\Delta^4 0^p$.	$\Delta^5 0^p$.	$\Delta^6 0^p$.
1	1					
2	1	2				
3	1	6	6			
4	1	14	36	24		
5	1	30	150	240	120	
6	1	62	540	1560	1800	720
7	1	126	1806	8400	16800	15120
8	1	254	5796	40824	126000	191520
9	1	510	18150	186480	834120	1905120
10	1	1022	55980	818520	5103000	16435440

From equation (2) we see that the value of a particular difference $\Delta^n 0^p$ is obtained by taking n times the sum of the two numbers of the preceding row which are situated in the same column and in the preceding column respectively. For example,

$$\begin{aligned}\Delta^3 0^7 &= 3(62+540)\\ &= 1806.\end{aligned}$$

6. The Differences of $x(x-1)(x-2)\ldots(x-p+1)$.—Among the polynomials of degree p there is one polynomial of special interest in the theory of interpolation, namely,

$$x(x-1)(x-2)\ldots(x-p+1).$$

This polynomial is denoted by $[x]^p$ and is called a *factorial*. If we suppose the interval of the argument in the difference table of $[x]^p$ to be unity, we have

$$[a]^p = a(a-1)(a-2) \;.\;.\;.\; (a-p+1),$$
$$[a+1]^p = (a+1)a(a-1)(a-2) \;.\;.\;.\; (a-p+2),$$
$$\Delta[a]^p = [a+1]^p - [a]^p$$
$$= a(a-1)(a-2)(a-3) \;.\;.\;.\; (a-p+2)\{(a+1)-(a-p+1)\}$$
$$= p[a]^{p-1},$$

so that
$$\Delta[x]^p = p[x]^{p-1}.*$$

It follows that

$$\frac{\Delta[x]^p}{p!} = \frac{[x]^{p-1}}{(p-1)!}, \text{ or } \frac{[x+1]^p}{p!} = \frac{[x]^p}{p!} + \frac{[x]^{p-1}}{(p-1)!},$$

a result that may now be used to tabulate the values of $[x]^p/p!$ as in the following table:

x.	$[x]^2/2!$.	$[x]^3/3!$.	$[x]^4/4!$.	$[x]^5/5!$.
0				
1	0			
2	1	0		
3	3	1	0	
4	6	4	1	0
5	10	10	5	1
6	15	20	15	6
7	21	35	35	21
8	28	56	70	56
9	36	84	126	126

7. **The Representation of a Polynomial by Factorials.**—In § 4 we found an expression for $\Delta f(x)$, the first difference of a polynomial of degree n, in a form which is less simple than the polynomial itself. It is more convenient to carry out the operation of differencing by the use of factorials, using the relation of § 6:

$$\Delta[x]^p = p[x]^{p-1}. \tag{1}$$

Let $\phi_k(x)$ denote a polynomial in x of degree k. We may write $\phi_k(x) = r + (x-n+k)\phi_{k-1}(x)$, where r is the remainder and $\phi_{k-1}(x)$ the quotient when $\phi_k(x)$ is divided by $(x-n+k)$, so $\phi_{k-1}(x)$ is of degree $(k-1)$. By a repeated application of this

* This is analogous to the formula of the differential calculus $\frac{d}{dx}(x^p) = px^{p-1}$.

transformation, we obtain an expression for a polynomial of the nth degree in terms of factorials:

$$\begin{aligned}\phi_n(x) &= \alpha + [x]\phi_{n-1}(x)\\ &= \alpha + \beta[x] + [x]^2\phi_{n-2}(x)\\ &= \alpha + \beta[x] + \gamma[x]^2 + [x]^3\phi_{n-3}(x)\\ &\quad\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot\\ &= \alpha + \beta[x] + \gamma[x]^2 + \ldots + [x]^n\phi_0(x),\end{aligned}$$

where $\alpha, \beta, \gamma, \ldots$ are constants and $\phi_0(x)$ is a constant ν (say). We thus obtain the result

$$\phi_n(x) = \alpha + \beta[x] + \gamma[x]^2 + \delta[x]^3 + \ldots + \nu[x]^n. \tag{2}$$

Ex.—To represent the function $y = x^4 - 12x^3 + 42x^2 - 30x + 9$ *and its successive differences in the factorial notation.*

Using detached coefficients when dividing by $x, x-1, x-2, \ldots,$*

$$\begin{array}{r|l}
1 & \begin{array}{l|l} 1 - 12 + 42 - 30 & 9 \\ 0 + 1 - 11 + 31 & \end{array} \\
2 & \begin{array}{l|l} 1 - 11 + 31 & 1 \\ 0 + 2 - 18 & \end{array} \\
3 & \begin{array}{l|l} 1 - 9 & 13 \\ 0 + 3 & \end{array} \\
\hline
1 & -6
\end{array}$$

we obtain the value of y in the form

$$y = [x]^4 - 6[x]^3 + 13[x]^2 + [x] + 9.$$

The successive differences are given by

$$\begin{aligned}\Delta y &= 4[x]^3 - 18[x]^2 + 26[x] + 1,\\ \Delta^2 y &= 12[x]^2 - 36[x] + 26,\\ \Delta^3 y &= 24[x] - 36,\\ \Delta^4 y &= 24.\end{aligned}$$

Now let a be one of the tabulated values of the argument of a polynomial of degree n, and let w be the interval between successive values of the argument. Consider the value $f(a+xw)$ of the polynomial corresponding to the value $(a+xw)$ of the argument. Writing $f(a+xw)$ for $\phi_n(x)$ in (2) and applying the operation denoted by equation (1) to both sides of equation (2), we find that

$$\Delta f(a+xw) = \beta + 2\gamma[x]^1 + 3\delta[x]^2 + \ldots + n\nu[x]^{n-1}. \tag{3}$$

* Chrystal, *Algebra*, **1**, p. 108.

Differencing this equation, we obtain

$$\Delta^2 f(a+xw)=2\gamma+2.3\delta[x]^1+3.4\epsilon[x]^2+\ldots+n(n-1)\nu[x]^{n-2}. \quad (4)$$

Moreover,

$$\Delta^3 f(a+xw)=2.3.\delta+2.3.4\epsilon[x]^1+3.4.5\xi[x]^2+\ldots$$
$$+n(n-1)(n-2)\nu[x]^{n-3}, \quad (5)$$

and so on for differences of higher order. The values of the coefficients $\alpha, \beta, \gamma, \ldots$ are found by putting $x=0$ in each of the equations (2), (3), (4), . . . so that

$$\alpha=f(a),\quad \beta=\Delta f(a),\quad \gamma=\tfrac{1}{2}\Delta^2 f(a),\quad \delta=\tfrac{1}{6}\Delta^3 f(a),\ \ldots\ \nu=\Delta^n f(a)/n!.$$

Equation (2) may now be written

$$f(a+xw)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\ldots$$
$$+\frac{x(x-1)(x-2)\ \ldots\ (x-n+1)}{n!}\Delta^n f(a).$$

This formula enables us to express the polynomial $f(a+xw)$ in terms of the factorials $x, x(x-1), x(x-1)(x-2), \ldots$ when a difference table of the function is given.*

This general formula may be easily verified for special values of x.

When $x=0$, it becomes $f(a)=f(a)$.

When $x=1$, then

$$f(a+w)=f(a)+1.\Delta f(a)$$
$$=f(a)+\{f(a+w)-f(a)\}, \text{ which is an identity.}$$

When $x=2$,

$$f(a+2w)=f(a)+2\Delta f(a)+\Delta^2 f(a)$$
$$=f(a)+2\{f(a+w)-f(a)\}$$
$$+\{f(a+2w)-2f(a+w)+f(a)\}.$$

8. **The Gregory-Newton Formula of Interpolation.**—The general formula of the last section may be applied to solve the problem of interpolation.

Suppose that y is a function of an argument u and that the values of y given in the table are $f(a)$, $f(a+w)$, $f(a+2w)$, $f(a+3w)$, . . . corresponding to the values a, $a+w$, $a+2w$, $a+3w$, . . . of u. Also suppose that these values of the function are entered in a difference table and that the differences of order n are constant. We are not supposed to know the values of y which correspond to other values of u, such as $u=a+\frac{1}{2}w$.

* Cf. Ex. 2, § 3.

It is required to find an analytical expression for these intermediate values of y.

The problem may be stated graphically as follows:

Draw the rectangular axes Ou, Oy. Let K, L, M, N . . . be points on the u axis having abscissae a, $a+w$, $a+2w$, $a+3w$, . . . respectively. At these points erect ordinates KA, LB, MC, ND, . . . equal respectively to the entries $f(a)$, $f(a+w)$, $f(a+2w)$, $f(a+3w)$, . . . Then the points A, B, C, D, . . . so determined are points on the graph of the function.* The problem of finding a "smooth" curve to pass through the points A, B, C, D, . . . has not a unique solution: in fact an infinite number of curves satisfying these conditions can be found. As our aim is a practical one, we naturally choose the simplest solution of our problem. Remembering that the simplest functions are polynomials, we inquire if it is possible to pass through the points A, B, C, . . . a curve which is the graph of a *polynomial* function of degree n.

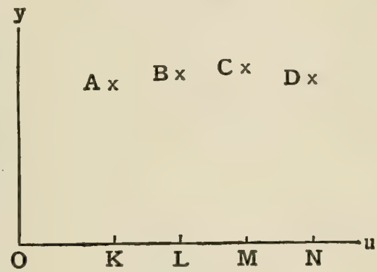

FIG. 1.

We have already seen (§ 4) that for any polynomial of degree n the differences of order n are constant and for the set of values $f(a)$, $f(a+w)$, $f(a+2w)$, . . . it has been assumed that the differences of order n are constant. This being so, a polynomial of degree n exists which takes the values $f(a)$, $f(a+w)$, $f(a+2w)$, . . . when the argument u has the values a, $a+w$, $a+2w$, . . .; in fact, by the last section, we can write down an expression for the polynomial. It is

$$y=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\ldots$$

$$+\frac{x(x-1)\ldots(x-n+1)}{n!}\Delta^n f(a) \quad (1)$$

$$=f(u),$$

* We do not know anything about the portions of the graph intermediate between these points, but we assume that the graph is a *smooth curve*; for our present purpose we can take this to mean that the function has finite differential coefficients of all orders at every point.

where x is connected with u by the relation $u=a+xw$, and where

$\Delta f(a)$ stands for $f(a+w)-f(a)$,
$\Delta^2 f(a)$ stands for $f(a+2w)-2f(a+w)+f(a)$,
and so on.

We shall now take the polynomial (1) *to represent the function y also for values of the argument intermediate between the tabulated values.* The portions of the graph intermediate between the points $A, B, C, \ldots$ may therefore be filled in by drawing the curve

$$y=f(a+xw)$$
$$=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\ldots \qquad (2)$$

and in order to compute the value of y corresponding to any intermediate value of the argument such as $a+\frac{1}{2}w$, we simply substitute the value $x=\frac{1}{2}$ in this formula,* which is the analytical expression required.

The fundamental problem of interpolation is thus solved. The formula (1) is often referred to as *Newton's formula of interpolation,* although it was discovered by James Gregory in 1670.†

The application of the Gregory-Newton formula is illustrated by the following examples :

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* Many books of logarithmic tables, etc., contain a table of the binomial coefficients required in the interpolation formula (1), at intervals of 0·01 from $x=0$ to $x=1$.

† Cf. a letter of Gregory to Collins of date November 23, 1670, printed in Rigaud's *Correspondence*, **2**, p. 209. An example of the use of the formula is worked out on p. 211 of Rigaud. Collins was accustomed to send on to Newton the mathematical discoveries of Gregory (cf. Rigaud, **2**, p. 335).

Newton's publications on interpolation are contained in :

1. The *Methodus Differentialis* published in 1711 but written before October 1676.
2. A letter written in 1676 to John Smith.
3. Lemma v. in Book iii. of the *Principia* published in 1687. The above formula is Case i.
4. Various references in the *Commercium Epistolicum* of dates 1672/3 to 1676.

These have been collected and edited by D. C. Fraser in the *Journal of the Institute of Actuaries*, **51** (1918–19), pp. 77 and 211.

Ex. 1.—*From the table given below to find the entry corresponding to* $x=21$.

Argument.	*Entry.*	Δ.	Δ^2.	Δ^3.	Δ^4.
20	0·229314955248				
		701747247			
22	0·230016702495		602297		
		702349544		−1944	
24	0·230719052039		600353		4
		702949897		−1940	
26	0·231422001936		598413		3
		703548310		−1937	
28	0·232125550246		596476		
		704144786			
30	0·232829695032				

Here $a=20$, $w=2$, $f(a+xw)=f(21)$, and $x=\frac{1}{2}$.

$$f(21)=f(20)+x\Delta f(20)+\frac{x(x-1)}{2}\Delta^2 f(20)+\frac{x(x-1)(x-2)}{3!}\Delta^3 f(20)+\ldots$$

$$\begin{aligned}
&=229314955248+\tfrac{1}{2}(701747247)-\tfrac{1}{8}(602297)-\tfrac{1}{16}(1944)\\
&=\begin{matrix}229314955248\\ +350873623\cdot5\end{matrix} \quad - \begin{cases}75287\cdot1\\ +121\cdot5\end{cases}\\
&=229665828871\cdot5-75408\cdot6
\end{aligned}$$

so

$f(21)=0\cdot229665753463$.

Ex. 2.—*To find the co-ordinate* X *of the sun on November* 10, 1910, *at* $4^h\ 30^m$ *G.M.T.* (X *is the sun's true geocentric co-ordinate measured on a line passing through the true equinox of the date*).

The *Nautical Almanac* gives the following readings from which we construct a difference table:

1910.	−X.	Δ.	Δ^2.	Δ^3.
November 9·0	0·6850997			
		−63809		
9·5	0·6787188		−514	
		−64323		4
10·0	0·6722865		−510	
		−64833		7
10·5	0·6658032		−503	
		−65336		2
11·0	0·6592696		−501	
		−65837		
11·5	0·6526859			

We must interpolate for $4^h\ 30^m$ from November 10·0. The argument is 12^h. Then $4^h\ 30^m$, as a fraction of the argument, gives $x=0\cdot375$.

$$\log x = 9{\cdot}5740313$$
$$\log (x-1) = 9{\cdot}7958800(n),$$

where (n) indicates that $9{\cdot}7958800$ is the logarithm of a negative number

$$\log \tfrac{1}{2} = 9{\cdot}6989700$$
$$\log \tfrac{1}{2}x(x-1) = 9{\cdot}0688813(n)$$
$$\log \tfrac{1}{3} = 9{\cdot}5228787$$
$$\log (x-2) = 0{\cdot}2108534(n)$$
$$\log \tfrac{1}{6}x(x-1)(x-2) = 8{\cdot}8026134.$$

Also

$$\log(-64833) = 4{\cdot}8117961(n) \qquad \log(-503) = 2{\cdot}7015680(n)$$
$$\log x = 9{\cdot}5740313 \qquad \log \tfrac{1}{2}x(x-1) = 9{\cdot}0688813(n)$$
$$\log(-64833x) = 4{\cdot}3858274(n) \qquad \log \tfrac{1}{2}x(x-1)(-503) = 1{\cdot}7704493$$
$$= \log(-24312{\cdot}4) \qquad = \log 58{\cdot}94$$

$$\log 2 = 0{\cdot}3010300$$
$$\log \tfrac{1}{6}x(x-1)(x-2) = 8{\cdot}8026134$$
$$\log \tfrac{1}{6}x(x-1)(x-2)(2) = 9{\cdot}1036434 = \log 0{\cdot}1.$$

Therefore $\quad -X = 0{\cdot}67228650 - 0{\cdot}00243124 + 0{\cdot}00000589,$

and finally $\quad -X = 0{\cdot}6698612.$

9. **An Alternative Form of the Gregory-Newton Formula.**—The Gregory-Newton formula may be written in an alternative form which is convenient when an arithmometer * is used. Rearranging the formula of the last section in the form

$$f(a+xw) = f(a) + x[\Delta f(a) - \tfrac{1}{2}(1-x)\{\Delta^2 f(a) - \tfrac{1}{3}(2-x)\,(\;.\;\;.\;\;.\;)\}],$$

and assuming the differences of order n to be constant, we may replace the Gregory-Newton formula by

$$f(a+xw) = f(a) + xu_1, \qquad (1)$$

where

$$u_1 = \Delta f(a) - \tfrac{1}{2}(1-x)u_2,$$
$$.\qquad .\qquad .\qquad .$$
$$u_p = \Delta^p f(a) - \frac{1}{p+1}(p-x)u_{p+1},$$
$$.\qquad .\qquad .\qquad .$$
$$u_n = \Delta^n f(a), \text{ which is constant.}$$

When computing a value of the function by this method, we begin with the constant difference u_n and calculate in succession the values of $u_{n-1}, u_{n-2}, \ldots, u_1$, finally substituting the value of u_1 in equation (1). The following example will serve as an illustration of this method:

* When an arithmometer is not available *Crelle's Calculating Tables* will be found useful for this purpose.

Ex.—*To find* $f(\theta)$ *when* $\theta = 24^\circ{\cdot}46980\ 05207\ 020$, *having given*

θ.	$f(\theta)$.	Δ.	Δ^2.	Δ^3.	Δ^4.
24·4	0·216 198 561 343				
		168 272 307			
24·5	0·216 366 833 650		745 715		
		169 018 022		768	
24·6	0·216 535 851 672		746 483		5
		169 764 505		773	
24·7	0·216 705 616 177		747 256		4
		170 511 761		777	
24·8	0·216 876 127 938		748 033		5
		171 259 794		782	
24·9	0·217 047 387 732		748 815		
		172 008 609			
25·0	0·217 219 396 341				

Here $w = 0{\cdot}1$, $a = 24^\circ{\cdot}4$, $x = 0{\cdot}698\ 005\ 207\ 02$.

Hence
$$u_3 = \Delta^3 f(a) - \tfrac{1}{4}(3 - x)\Delta^4 f(a) = 768 - 0{\cdot}576 \times 5$$
$$= 765{\cdot}1,$$
$$u_2 = \Delta^2 f(a) - \tfrac{1}{3}(2 - x)u_3 = 745\ 715 - 0{\cdot}434\ 0 \times 765{\cdot}1$$
$$= 745\ 383{\cdot}0,$$
$$u_1 = \Delta f(a) - \tfrac{1}{2}(1 - x)u_2 = 168\ 272\ 307 - 0{\cdot}150\ 997\ 4 \times 745\ 383$$
$$= 168\ 159\ 756{\cdot}1.$$

Then
$$f(a + xw) = f(a) + xu_1$$
$$= 0{\cdot}216\ 198\ 561\ 343 + 0{\cdot}698\ 005\ 207\ 02 \times 0{\cdot}000\ 168\ 159\ 756$$
$$= 0{\cdot}216\ 198\ 561\ 343$$
$$+ 117\ 376\ 385,$$

or $f(\theta) = 0{\cdot}216\ 315\ 937\ 728.$

10. **The Binomial Theorem.**—By use of the operator E, we can write the Gregory-Newton interpolation formula in the form

$$E^x f(a) = \{1 + x\Delta + \frac{x(x-1)}{2!}\Delta^2 + .\ .\ .\}f(a).$$

When thus written, the formula is seen to be the same as that obtained by expanding $(1 + \Delta)^x$ by the Binomial Theorem in ascending powers of Δ and then operating on $f(a)$ with the terms of the series so formed, *i.e.*

$$E^x f(a) = (1 + \Delta)^x f(a).$$

The Binomial Theorem was made known (in correspondence) six years after the Gregory-Newton formula; in fact, Newton seems to have discovered the Binomial Theorem by forming the expansions of $(1 + x)^n$ directly for integral values of n, and then writing down the powers of x in these expansions. In the case of the coefficient of x^2 he would have:

Exponent.	*Coefficient of x^2.*	Δ.	Δ^2.
0	0		
		0	
1	0		
		1	
2	1		1
		2	
3	3		1
		3	
4	6		1
		4	
5	10		

whence evidently the coefficient is of the second degree in n. Since it vanishes when $n=0$ and also when $n=1$, it must contain the factors n and $(n-1)$; and, since the coefficient has the value 1 when $n=2$, it is $\frac{n(n-1)}{2}$.

We may remark that if we form a difference table for $(1+x)^n$ thus:

Argument.	*Entry.*	Δ.	Δ^2.	Δ^3.
0	1			
		x		
1	$(1+x)^1$		x^2	
		$x(1+x)$		x^3
2	$(1+x)^2$		$x^2(1+x)$	
		$x(1+x)^2$		$x^3(1+x)$
3	$(1+x)^3$		$x^2(1+x)^2$	

then on substituting the values $f(0)=1$, $\Delta f(0)=x$ . . . in the Gregory-Newton formula

$$f(n)=f(0)+n\Delta f(0)+\tfrac{1}{2}n(n-1)\Delta^2 f(0)+\ldots$$

we obtain
$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

which is the binomial expansion.

Examples on Chapter I

1. Form the difference tables corresponding to the following entries:

	θ.	*log tan θ.*
(*a*)	26° 10′ 0″	9·691 380 858 103 01
	10″	434 054 052 28
	20″	487 246 020 72
	30″	540 434 009 42
	40″	593 618 019 47
	50″	646 798 051 97
	26° 11′ 0″	9·691 699 974 108 01
	10″	753 146 188 70
	20″	806 314 295 11
	30″	859 478 428 36

	x.	$sin\ x$.
(b)	28° 40′ 00″	0·479 713 113 250 246
	10″	755 651 470 168
	20″	798 188 562 452
	30″	840 724 526 998
	40″	883 259 363 705
	50″	925 793 072 474
	28° 41′ 00″	968 325 653 205
	10″	0·480 010 857 105 798

2. If $y = 2x^3 - x^2 + 3x + 1$, calculate the values of y corresponding to $x = 0, 1, 2, 3, 4, 5$, and form the table of differences. Prove theoretically that the second difference is $12x + 10$ and verify this numerically.

3. Find the function whose first difference is the function

$$\alpha x^3 + \beta x^2 + \gamma x + \delta.$$

4. Find the successive differences of

(a) $1/x$, the interval being unity,
(b) $\cos nx$, the interval being w.

5. Express $f(x) = 3x^3 + x^2 + x + 1$ in the form

$$\alpha x(x-1)(x-2) + \beta x(x-1) + \gamma x + \delta$$

by comparing coefficients. Calculate the values of $f(x)$ for $x = 0, 1, 2, 3, 4, 5$, etc., and form a difference table. Verify the equation

$$f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0).$$

6. Compute the third difference of $f(51)$ by the formula of § 3, Ex. 1, from the following table of entries:

x	51	52	53	54
$f(x)$	132651	140608	148877	157464

verifying the result by means of a difference table.

7. Given the table of values

x	− 3	− 2	− 1	0	1
y	16	7	4	1	− 8

find by means of the Gregory-Newton formula an expression for y as a function of x.

8. Construct a difference table having given

log 5·950 = 0·776 701 184 0
log 5·951 = 0·776 773 802 4
log 5·952 = 0·776 846 408 7
log 5·953 = 0·776 919 002 8
log 5·954 = 0·776 991 584 9

and determine log 5·9505.

9. Let p, q, r, s be successive entries in a table corresponding to equidistant arguments.

Show that when third differences are taken into account the entry corresponding to the argument half-way between the arguments of q and r is

$$\frac{q+r}{2}+\frac{(q+r)-(p+s)}{16}. \qquad \text{(De Morgan.)}$$

10. Let p, q, r, s be successive entries (corresponding to equidistant arguments) in a table. It is required to interpose 3 entries (corresponding to equidistant arguments) between q and r, using third differences. Show that this may be done as follows:

Between q and r interpose 3 arithmetical means A, B, and C; also between $3q-2p-s$ and $3r-2s-p$ interpose 3 means A′, B′, and C′. Then the 3 terms required are $A+\frac{1}{32}A'$, $B+\frac{1}{24}B'$, $C+\frac{1}{32}C'$.

(De Morgan.)

11. Determine log 6·0405, having given

$$\begin{aligned}
\log 6{\cdot}040 &= 0{\cdot}7810369386\\
\log 6{\cdot}041 &= 0{\cdot}7811088357\\
\log 6{\cdot}042 &= 0{\cdot}7811807209\\
\log 6{\cdot}043 &= 0{\cdot}7812525942\\
\log 6{\cdot}044 &= 0{\cdot}7813244557
\end{aligned}$$

12. Using the method of § 9, find sin 24°·4698005207, having given the values

θ.	*sin* θ.
24·25	0·410718852614
24·50	0·414693242656
24·75	0·418659737537
25·00	0·422618261741
25·25	0·426568739902
25·50	0·430511096808

13. Given the values

x.	$f(x)$.
0	858·313740095
1	869·645772308
2	880·975826766
3	892·303904583
4	903·630006875

calculate $f(1{\cdot}5)$ by the Gregory-Newton formula.

14. The values of a function corresponding to the values 1, 2, 3, 4, 5 of the argument are 0·198669, 0·237702, 0·276355, 0·314566, 0·352274 respectively. Calculate the values of the function when the argument has the values 1·25 and 1·75 respectively.

15. Using the difference table given in § 2, find the values of sin 25° 40′ 10″ and sin 25° 40′ 30″. Also verify the answers

$$\begin{aligned}
\sin 25^\circ\, 40'\, 50'' &= 0{\cdot}433\;353\;261\;493\;416,\\
\sin 25^\circ\, 41'\, 10'' &= 0{\cdot}433\;440\;644\;614\;711,\\
\sin 25^\circ\, 41'\, 30'' &= 0{\cdot}433\;528\;023\;660\;896,\\
\sin 25^\circ\, 41'\, 50'' &= 0{\cdot}433\;615\;398\;631\;149,
\end{aligned}$$

obtained by taking x numerically less than unity in the formula of § 8.

16. Calculate log tan 24° 0′ 5″, given the values

log tan 24° 0′ 0″ = 9·648 583 137 400 95
log tan 24° 0′ 20″ = 9·648 696 457 723 08
log tan 24° 0′ 40″ = 9·648 809 758 267 66
log tan 24° 1′ 0″ = 9·648 923 039 045 83
log tan 24° 1′ 20″ = 9·649 036 300 068 75
log tan 24° 1′ 40″ = 9·649 149 541 347 57.

17. The following table gives the values of $I(x) = \int_x^\infty e^{-s^2} ds$:

x.	$I(x)$.
0·00	0·886 226 92
0·01	0·876 227 24
0·02	0·866 229 57
0·03	0·856 235 90
0·04	0·846 248 22
0·05	0·836 268 53

Calculate $I(x)$ for $x = 0{\cdot}025$ by interpolation and verify your result by use of the formula

$$I(0) - I(x) = x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \ldots$$

CHAPTER II

INTERPOLATION WITH UNEQUAL INTERVALS OF THE ARGUMENT

11. **Divided Differences.**—We have so far assumed that the values of the argument proceed by equal steps; but with data derived from observation it is not always possible to complete a difference table in this way. For example, when astronomical observations are disturbed by clouds there are gaps in the records.

Consider the case in which the values of the argument, for which the function is known, are unequally spaced, and suppose that the values of $f(x)$ are known for $x=a_0$, $x=a_1$, $x=a_2$, . . ., $x=a_n$, where the intervals a_1-a_0, a_2-a_1, a_3-a_2, . . ., a_n-a_{n-1} need not be equal. In place of ordinary differences we now introduce what are known as *divided differences.** Let us form in succession the quantities

$$\frac{f(a_1)-f(a_0)}{a_1-a_0}=f(a_1,a_0), \frac{f(a_2)-f(a_1)}{a_2-a_1}=f(a_2,a_1), \frac{f(a_3)-f(a_2)}{a_3-a_2}=f(a_3,a_2),$$

and so on. These are called *divided differences of the first order.* Moreover, let us form

$$\frac{f(a_2,a_1)-f(a_1,a_0)}{a_2-a_0}=f(a_2,a_1,a_0), \frac{f(a_3,a_2)-f(a_2,a_1)}{a_3-a_1}=f(a_3,a_2,a_1).$$

These are called *divided differences of the second order.* Also let

$$\{f(a_3,a_2,a_1)-f(a_2,a_1,a_0)\}/(a_3-a_0)=f(a_3,a_2,a_1,a_0).$$

This is called a *divided difference of the third order.* The divided differences of higher orders are formed in the same way, so that the order of a divided difference is less by unity than the number of arguments required for its definition.

* Divided differences might fairly be ascribed to Newton, Lemma v. The term was used first by De Morgan, *Diff. and Int. Calc.* (1842), p. 550, and afterwards by Oppermann, *Journ. Inst. Act.* **15** (1869), p. 146. Ampère, *Ann. de Gergonne*, **26** (1826), p. 329, used the name *interpolatory functions.*

Divided differences may be expressed more symmetrically as follows:

$$f(a_1, a_0) = \frac{f(a_0)}{a_0 - a_1} + \frac{f(a_1)}{a_1 - a_0},$$

$$\begin{aligned} f(a_2, a_1, a_0) &= \frac{1}{a_2 - a_0}\left\{\frac{f(a_2)}{a_2 - a_1} + \frac{f(a_1)}{a_1 - a_2}\right\} + \frac{1}{a_0 - a_2}\left\{\frac{f(a_1)}{a_1 - a_0} + \frac{f(a_0)}{a_0 - a_1}\right\} \\ &= \frac{f(a_0)}{(a_0 - a_1)(a_0 - a_2)} + \frac{f(a_1)}{(a_1 - a_0)(a_1 - a_2)} + \frac{f(a_2)}{(a_2 - a_1)(a_2 - a_0)}, \end{aligned}$$

$$\begin{aligned} f(a_3, a_2, a_1, a_0) &= \frac{f(a_0)}{(a_0 - a_1)(a_0 - a_2)(a_0 - a_3)} + \frac{f(a_1)}{(a_1 - a_0)(a_1 - a_2)(a_1 - a_3)} \\ &\quad + \frac{f(a_2)}{(a_2 - a_0)(a_2 - a_1)(a_2 - a_3)} + \frac{f(a_3)}{(a_3 - a_0)(a_3 - a_1)(a_3 - a_2)}. \end{aligned}$$

In general, as may easily be shown by induction, a divided difference of the pth order is a symmetric function of its arguments and is in fact the sum of $(p+1)$ functions of the form

$$\frac{f(a_r)}{\text{difference-product of } a_r \text{ with } a_0, a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_p}$$

It is evident from this statement that when the arguments required to form a particular divided difference are arranged in a different order, the value of the divided difference remains unchanged, *e.g.*

$$f(a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0) = f(a_0, a_1, a_2, \ldots, a_{n-1}, a_n).$$

Divided differences are arranged in a *table of divided differences* as follows:

Argument.	*Entry.*			
a_0	$f(a_0)$			
		$f(a_0, a_1)$		
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		$f(a_1, a_2, a_3, a_4)$
a_3	$f(a_3)$		$f(a_2, a_3, a_4)$	
		$f(a_3, a_4)$		$f(a_2, a_3, a_4, a_5)$
a_4	$f(a_4)$		$f(a_3, a_4, a_5)$	

The following may serve as an example of a table of divided differences :

x.	$f(x)$.			
0	132651			
		8113		
2	148877		158	
		8587		1
3	157464		162	
		8911		1
4	166375		167	
		9579		1
7	195112		173	
		10444		
9	216000			

In this example the differences of the third order are constant. We shall now see under what circumstances a column of constant divided differences is obtained.

12. Theorems on Divided Differences.

I. *If a function $f(x)$ is numerically equal to the sum of two functions $g(x)$, $h(x)$, for a set of values of the argument x, then any divided difference of $f(x)$ formed from those values is equal to the sum of the corresponding divided differences of $g(x)$ and $h(x)$.*

For example,

$$f(a_1, a_0) = \frac{f(a_1) - f(a_0)}{a_1 - a_0} = \frac{\{g(a_1) - g(a_0)\} + \{h(a_1) - h(a_0)\}}{a_1 - a_0}$$

$$= g(a_1, a_0) + h(a_1, a_0),$$

and similarly for differences of higher order.

II. *A divided difference of $cf(x)$, where c is a constant factor, is c times the corresponding divided difference of $f(x)$.*

For example, the divided difference of the first order of $cf(x)$ is

$$\frac{cf(a_1) - cf(a_0)}{a_1 - a_0} = c\frac{f(a_1) - f(a_0)}{a_1 - a_0} = cf(a_1, a_0).$$

III. *The divided differences of order n of x^n are constant (where n is a positive integer).*

Let $$f(x) = x^n.$$

Then $$f(a_0, a_1) = (a_0^n - a_1^n)/(a_0 - a_1)$$

$$= a_0^{n-1} + a_1 a_0^{n-2} + \ldots + a_1^{n-1}$$

a homogeneous function of a_0, a_1 of degree $(n-1)$. Moreover,

$$f(a_0, a_1, a_2)$$
$$= \frac{[a_0^{n-1} + a_1 a_0^{n-2} + \ldots + a_1^{n-1}] - [a_2^{n-1} + a_1 a_2^{n-2} + \ldots + a_1^{n-1}]}{a_0 - a_2}$$
$$= (a_0^{n-1} - a_2^{n-1})/(a_0 - a_2) + a_1(a_0^{n-2} - a_2^{n-2})/(a_0 - a_2) + \ldots$$
$$+ a_1^{n-2}(a_0 - a_2)/(a_0 - a_2)$$
$$= (a_0^{n-2} + a_2 a_0^{n-3} + \ldots + a_2^{n-2})$$
$$+ a_1(a_0^{n-3} + a_2 a_0^{n-4} + \ldots + a_2^{n-3}) + \ldots$$

which is a homogeneous function of a_0, a_1, a_2 of degree $(n-2)$. In general $f(a_0, a_1, a_2, \ldots, a_p)$ is a homogeneous function of $a_0, a_1, a_2, \ldots, a_p$ of degree $(n-p)$. Taking $p = n$, we see that $f(a_0, a_1, a_2, \ldots, a_n)$ is a constant.

Corollary: The divided differences of order $(n+1)$ *of* x^n *are zero.*

IV. *The divided differences of order n of a polynomial of the nth degree are constant.*

This theorem follows immediately from theorems I., II., and III., since the divided difference of order n of each of the terms whose degree is less than n is zero.

V. *A divided difference of order r may be expressed as the quotient of two determinants each of order* $r+1$.

Consider the divided difference of the third order,

$$f(a_0, a_1, a_2, a_3) = \Sigma \frac{f(a_0)}{(a_0 - a_1)(a_0 - a_2)(a_0 - a_3)}$$
$$= \Sigma \frac{f(a_0)\ (\text{difference-product of } a_1, a_2, a_3)}{\text{difference-product of } a_0, a_1, a_2, a_3}.$$

Now a difference-product may be expressed as a determinant of the kind known as Vandermonde's, thus

$$(\text{difference-product of } a_1, a_2, a_3) = \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1 & a_2 & a_3 \\ 1 & 1 & 1 \end{vmatrix}$$

Therefore

$$f(a_0, a_1, a_2, a_3) = \Sigma \frac{f(a_0) \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ a_1 & a_2 & a_3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0 & a_1 & a_2 & a_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}$$

or

$$f(a_0, a_1, a_2, a_3) = \frac{\begin{vmatrix} f(a_0) & f(a_1) & f(a_2) & f(a_3) \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0 & a_1 & a_2 & a_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0 & a_1 & a_2 & a_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}}$$

and so in general for differences of order higher than the third.

13. **Newton's Formula for Unequal Intervals.**—Let $f(u)$ be a function whose divided differences of (say) order 4 vanish or are negligible; and suppose its values for 4 arguments a_0, a_1, a_2, a_3 are known so that the table of divided differences is as follows:

Argument.	*Entry.*			
a_0	$f(a_0)$			
		$f(a_0, a_1)$		
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		$f(a_1, a_2, a_3, a_4)$
a_3	$f(a_3)$			

We may obtain the value of the function for any other argument u in the following way. Beginning with the constant difference which is of order 3, we have

$$f(u, a_0, a_1, a_2) = f(a_0, a_1, a_2, a_3). \tag{1}$$

By definition of the divided difference of order 2,

$$f(u, a_0, a_1) = f(a_0, a_1, a_2) + (u - a_2) f(u, a_0, a_1, a_2),$$

and therefore

$$f(u, a_0, a_1) = f(a_0, a_1, a_2) + (u - a_2) f(a_0, a_1, a_2, a_3). \tag{2}$$

Again by definition,

$$f(u, a_0) = f(a_0, a_1) + (u - a_1) f(u, a_0, a_1), \tag{3}$$

and substituting in this equation the value of $f(u, a_0, a_1)$ from (2),

$$f(u, a_0) = f(a_0, a_1) + (u - a_1) f(a_0, a_1, a_2) + (u - a_1)(u - a_2) f(a_0, a_1, a_2, a_3).$$

Also by definition $\quad f(u) = f(a_0) + (u - a_0) f(u, a_0), \quad$ (4)

or $\quad f(u) = f(a_0) + (u - a_0) f(a_0, a_1) + (u - a_0)(u - a_1) f(a_0, a_1, a_2)$

$$+ (u - a_0)(u - a_1)(u - a_2) f(a_0, a_1, a_2, a_3). \tag{5}$$

From the equations (1), (2), (3), (4) the quantities $f(u, a_0, a_1, a_2)$, $f(u, a_0, a_1)$, $f(u, a_0)$, $f(u)$ are now known and may be inserted in the table of divided differences thus:*

Argument.	*Entry.*			
u	$f(u)$			
		$f(u, a_0)$		
a_0	$f(a_0)$		$f(u, a_0, a_1)$	
		$f(a_0, a_1)$		$f(u, a_0, a_1, a_2)$
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		
a_3	$f(a_3)$			

Formula (5) may evidently be generalised to express a function whose divided differences of order $(n+1)$ are negligible or zero, in the form

$$\begin{aligned} f(u) = f(a_0) &+ (u-a_0)f(a_0, a_1) + (u-a_0)(u-a_1)f(a_0, a_1, a_2) \\ &+ (u-a_0)(u-a_1)(u-a_2)f(a_0, a_1, a_2, a_3) + \ldots \\ &+ (u-a_0)(u-a_1)\ldots(u-a_{n-1})f(a_0, a_1, \ldots, a_n). \qquad (6) \end{aligned}$$

This formula was discovered by Newton.†

The first term on the right-hand side of this equation represents the polynomial of zero degree, which has the value $f(a_0)$ at $u = a_0$. The first two terms together represent the polynomial of degree 1, which has the values $f(a_0)$ and $f(a_1)$ at a_0 and a_1 respectively, and so on.

The remainder term which must be added to the right-hand side of the equation in order to obtain strict accuracy is in fact

$$(u-a_0)(u-a_1)\ \ldots\ (u-a_n)f(u, a_0, a_1, \ldots, a_n).$$

But this term vanishes if the divided differences of order n are rigorously constant.

Ex.—From the table given below to find the entry corresponding to 3·7608.

x.	$f(x)$.		
$a_0 = 0$	·3989423		
		− 500	
$a_1 = 2{\cdot}5069$	·3988169		− 199
		− 1499	
$a_2 = 5{\cdot}0154$	·3984408		− 199
		− 2496	
$a_3 = 7{\cdot}5270$	·3978138		

* In practice the value of $f(u)$ is usually found by forming the successive divided differences in this way, as in the worked-out example below.

† *Principia* (1687), Book iii. Lemma v. Case ii. Cf. Cauchy, *Œuvres*, (1) 5, p. 409.

Forming the successive divided differences of $f(u)$, where $u = 3{\cdot}7608$, we find

$$f(u, a_0, a_1) = f(a_0, a_1, a_2) = -199,$$
$$f(u, a_0) = -500 + 1{\cdot}2539 \times (-199) = -749{\cdot}526,$$
$$f(u) = {\cdot}3989423 + 3{\cdot}7608 \times (-749{\cdot}526).$$

The calculated value is therefore 0·3986604.

14. **The Gregory-Newton Formula as a Special Case of Newton's Formula.**—The Gregory-Newton formula may be regarded as the special case of the formula of the last section when the intervals of the argument are equal.

For in Newton's formula for unequal intervals suppose that we put

$$a_0 = a, \quad a_1 = a + w, \quad a_2 = a + 2w, \ . \ . \ ., \ u = a + xw.$$

By constructing a table of divided differences, we see that

$$f(a_0, a_1) = \frac{1}{w}\Delta f(a), \quad f(a_1, a_2) = \frac{1}{w}\Delta f(a + w),$$

$$\therefore f(a_0, a_1, a_2) = \frac{1}{2!\,w^2}\Delta^2 f(a).$$

In the same way we find

$$f(a_0, a_1, a_2, a_3) = \frac{1}{3!\,w^3}\Delta^3 f(a),$$

and so on.

If we now replace u by $a + xw$, the formula for unequal intervals of the argument becomes

$$f(a + xw) = f(a) + x\Delta f(a) + \frac{x(x-1)}{2!}\Delta^2 f(a) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(a) + \ . \ . \ .$$

which is the Gregory-Newton formula.

15. **The Practical Application of Newton's Formula.**—In laboratory computation from Newton's formula, we proceed by a method which is really identical with that given above (Ex. § 13). Rearranging the formula of § 13, we see that

$$f(u) = f(a_0) + (u - a_0)\,[f(a_0, a_1)$$
$$+ (u - a_1)\{f(a_0, a_1, a_2) + (u - a_2)(f(a_0, a_1, a_2, a_3) + \ . \ . \ .)\}].$$

This equation may be written in the form

$$f(u) = f(a_0) + (u - a_0)v_1, \qquad (1)$$

where
$$\begin{cases} v_1 = f(a_0, a_1) + (u - a_1)v_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_r = r\text{th divided difference} + (u - a_r)v_{r+1}, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_n = f(a_0, a_1, \ . \ . \ ., a_n), \text{ a constant.} \end{cases}$$

The v's are computed in the following order: $v_{n-1}, v_{n-2}, \ . \ . \ ., v_1$. The value of $f(u)$ is then obtained from equation (1).

Ex.—To find the function corresponding to the argument 6·417 in the following difference table:

Argument.	*Entry.*			
$a_0=5$	150			
		121		
$a_1=7$	392		24	
		265		1
$a_2=11$	1452		32	
		457		1
$a_3=13$	2366		46	
		917		
$a_4=21$	9702			

$$u=6{\cdot}417,\quad v_3=1,\quad v_2=24+(6{\cdot}417-11)1=19{\cdot}417,$$
$$v_1=121\times(6{\cdot}417-7)19{\cdot}417=109{\cdot}679889,$$
$$\therefore\ f(6{\cdot}417)=150+(6{\cdot}417-5)109{\cdot}679889$$
$$=305{\cdot}416402713.$$

16. Divided Differences with Repeated Arguments.—The original definition of divided differences presupposes that the arguments concerned are all different. If, however, the quantity $f(a_0, a_0+\epsilon)$ tends to a definite limit as ϵ tends to zero, we denote this limit by $f(a_0, a_0)$, and similarly for divided differences of higher order.

Now suppose that in § 13, $u=a_0$. Since the differences of order 3 are supposed constant, we see that $f(a_0, a_0, a_1, a_2)$ is equal to $f(a_0, a_1, a_2, a_3)$, and the remaining differences $f(a_0, a_0, a_1), f(a_0, a_0)$ may then be calculated just as in the general case when u and a_0 were supposed different. We may now form another set of differences by again taking $u=a_0$. Repeating this method, we obtain the following table of divided differences:

Argument.	*Entry.*			
a_0	$f(a_0)$			
		$f(a_0, a_0)$		$f(a_0, a_0, a_0, a_0)$
a_0	$f(a_0)$		$f(a_0, a_0, a_0)$	
		$f(a_0, a_0)$		$f(a_0, a_0, a_0, a_1)$
a_0	$f(a_0)$		$f(a_0, a_0, a_1)$	
		$f(a_0, a_1)$		$f(a_0, a_0, a_1, a_2)$
a_1	$f(a_1)$		$f(a_0, a_1, a_2)$	
		$f(a_1, a_2)$		$f(a_0, a_1, a_2, a_3)$
a_2	$f(a_2)$		$f(a_1, a_2, a_3)$	
		$f(a_2, a_3)$		
a_3	$f(a_3)$			

In terms of these divided differences with repeated arguments the formula of Newton becomes

$$f(u) = f(a_0) + (u - a_0)f(a_0, a_0) + (u - a_0)^2 f(a_0, a_0, a_0) + (u - a_0)^3 f(a_0, a_0, a_0, a_0) + \ldots$$

This formula will be used later to obtain an expression for the derivatives of a function in terms of its divided differences.*

Ex.—Given the values $\begin{matrix} x & 5 & 11 & 27 & 34 & 42 \\ f(x) & 23 & 899 & 17315 & 35606 & 68510 \end{matrix}$ *to find* $f(x)$ *in terms of powers of* $(x-3)$.

Constructing a table of divided differences and extending it to include repeated arguments for $x = 3$, we obtain

x.	$f(x)$.			
42	68510			
		4113		
34	35606		100	
		2613		1
27	17315		69	
		1026		1
11	899		40	
		146		1
5	23		16	
		18		1
3	− 13		8	
		2		1
3	− 13		6	
		2		
3	− 13			

Applying Newton's formula for repeated arguments, the required value is $f(x) = -13 + 2(x-3) + 6(x-3)^2 + (x-3)^3$.

17. **Lagrange's Formula of Interpolation.**—Let $f(x)$ be the polynomial of degree n which for values $a_0, a_1, a_2, \ldots, a_n$ of the argument x has the values $f(a_0), f(a_1), \ldots, f(a_n)$ respectively. By the definition of divided differences, we have

$$f(a_0, a_1, a_2, \ldots, a_n, x) = \frac{f(x)}{(x-a_0)(x-a_1)\ldots(x-a_n)} + \frac{f(a_0)}{(a_0-x)(a_0-a_1)\ldots(a_0-a_n)} + \frac{f(a_1)}{(a_1-x)(a_1-a_0)\ldots(a_1-a_n)} + \ldots + \frac{f(a_n)}{(a_n-x)(a_n-a_0)\ldots(a_n-a_{n-1})}.$$

* § 37.

Since $f(x)$ is a polynomial of degree n, its divided differences of order $(n+1)$ are zero, *i.e.*

$$f(a_0, a_1\, a_2, \ldots a_n, x) = 0.$$

Arranging the factors of the denominators in the above fractions so that the first factor in each denominator is of the form $(x - a_p)$, we obtain

$$\begin{aligned}\frac{f(x)}{(x-a_0)(x-a_1)\ldots(x-a_n)} &= \frac{f(a_0)}{(x-a_0)(a_0-a_1)(a_0-a_2)\ldots(a_0-a_n)} \\ &+ \frac{f(a_1)}{(x-a_1)(a_1-a_0)\ldots(a_1-a_n)} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ \frac{f(a_n)}{(x-a_n)(a_n-a_0)\ldots(a_n-a_{n-1})}, \qquad \text{(A)}\end{aligned}$$

which is Lagrange's formula in a form suitable for computation.*

Another way of writing this formula is obtained by multiplying both sides of equation (A) by

$$(x-a_0)(x-a_1)(x-a_2)\ldots(x-a_n),$$

when we obtain

$$\begin{aligned}f(x) &= \frac{(x-a_1)(x-a_2)\ldots(x-a_n)}{(a_0-a_1)(a_0-a_2)\ldots(a_0-a_n)}f(a_0) \\ &+ \frac{(x-a_0)(x-a_2)\ldots(x-a_n)}{(a_1-a_0)(a_1-a_2)\ldots(a_1-a_n)}f(a_1) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ \frac{(x-a_0)(x-a_1)\ldots(x-a_{n-1})}{(a_n-a_0)(a_n-a_1)\ldots(a_n-a_{n-1})}f(a_n). \qquad \text{(B)}\end{aligned}$$

It is important to note that when a set of experimental data obey a law which can be expressed algebraically as a polynomial of degree n, then not less than $(n+1)$ observations are required in order to construct the polynomial. If only n values were used, the resulting polynomial would be of degree $(n-1)$. Before applying the Lagrange formula it is therefore necessary to ascertain the order of the divided differences which are of constant value and thus find the proper value for n.

Ex. 1.—*Given the values*

x	14	17	31	35
$f(x)$	68·7	64·0	44·0	39·1

to calculate the value of $f(x)$ *corresponding to* $x = 27$.

* Lagrange's formula was first published in his *Leçons élémentaires sur les mathématiques*, in 1795, reprinted in his *Œuvres*, 7, p. 286.

Applying formula (A), we obtain

$$\frac{f(27)}{(27-14)(27-17)(27-31)(27-35)}$$

$$=\frac{68\cdot7}{(27-14)(14-17)(14-31)(14-35)}$$

$$+\frac{64\cdot0}{(27-17)(17-14)(17-31)(17-35)}$$

$$+\frac{44\cdot0}{(27-31)(31-14)(31-17)(31-35)}$$

$$+\frac{39\cdot1}{(27-35)(35-14)(35-17)(35-31)}$$

or

$$\frac{f(27)}{4160}=-\frac{68\cdot7}{13923}+\frac{64\cdot0}{7560}+\frac{44\cdot0}{3808}-\frac{39\cdot1}{12096},$$

$$\therefore f(27)=49\cdot317 \text{ (approx.)}.$$

The required value is 49·3.

Ex. 2.—*Given the data* $\begin{matrix} x & 0 & 1 & 2 & 5 \\ f(x) & 2 & 3 & 12 & 147 \end{matrix}$, *to form the cubic function of* x.

Applying formula (B), we have

$$f(x)=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}2+\frac{x(x-2)(x-5)}{1(1-2)(1-5)}3+\frac{x(x-1)(x-5)}{2(2-1)(2-5)}12$$
$$+\frac{x(x-1)(x-2)}{5(5-1)(5-2)}147$$
$$=x^3+x^2-x+2.$$

18. **An alternative proof of Lagrange's formula by the use of determinants** is the following:

Let P_n denote a polynomial of degree n, and put

$$P_n=A+Bx+Cx^2+\ldots+Lx^n$$
$$=f(x).$$

Substituting in succession the values $a_0, a_1, \ldots, a_n$ for x, we obtain

$$f(a_0)=A+Ba_0+Ca_0^2+\ldots+La_0^n,$$
$$f(a_1)=A+Ba_1+Ca_1^2+\ldots+La_1^n,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$f(a_n)=A+Ba_n+Ca_n^2+\ldots+La_n^n.$$

Eliminating A, B, C, . . . from these equations determinantally we have

$$0 = \begin{vmatrix} P_n & f(a_0) & f(a_1) & f(a_2) & \dots & f(a_n) \\ 1 & 1 & 1 & 1 & \dots & 1 \\ x & a_0 & a_1 & a_2 & \dots & a_n \\ x^2 & a_0^2 & a_1^2 & a_2^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^n & a_0^n & a_1^n & a_2^n & \dots & a_n^n \end{vmatrix}$$

Expanding this determinant according to the elements of the first row, we see that

$$P_n \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ a_0^2 & a_1^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} = f(a_0) \begin{vmatrix} 1 & 1 & \dots & 1 \\ x & a_1 & \dots & a_n \\ x^2 & a_1^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot \\ x^n & a_1^n & \dots & a_n^n \end{vmatrix}$$

$$-f(a_1) \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x & a_0 & a_2 & \dots & a_n \\ x^2 & a_0^2 & a_2^2 & \dots & a_n^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^n & a_0^n & a_2^n & \dots & a_n^n \end{vmatrix} + \dots + (-1)^n f(a_n) \begin{vmatrix} 1 & 1 & \dots & 1 \\ x & a_0 & \dots & a_{n-1} \\ x^2 & a_0^2 & \dots & a^2_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ x^n & a_0^n & \dots & a^n_{n-1} \end{vmatrix} \quad (1)$$

The determinants in this equation may be represented as difference-products. The coefficient of $f(a_0)$ is the difference-product of $x, a_1, \dots, a_n$, the coefficient of $f(a_1)$ is the difference-product of $x, a_0, a_2, \dots, a_n$, and so on. We may write

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot \\ a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_0 & a_2 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1^n & a_0^n & a_2^n & \dots & a_n^n \end{vmatrix} = + \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_2 & a_0 & a_1 & \dots & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_2^n & a_0^n & a_1^n & \dots & a_n^n \end{vmatrix} = - \dots$$

i.e. the coefficient of P_n is equal to the difference-product of $a_0, a_1, \dots, a_n$: it is also equal to minus the difference-product of $a_1, a_0, a_2, \dots, a_n$, or to plus the difference-product of $a_2, a_0, a_1, \dots, a_n$, and so on. If we now divide throughout by the coefficient of P_n in equation (1), we obtain the result:

$$\begin{aligned}P_n = f(a_0)&\frac{(x-a_1)(x-a_2)(x-a_3)\ .\ .\ .\ (x-a_n)}{(a_0-a_1)(a_0-a_2)(a_0-a_3)\ .\ .\ .\ (a_0-a_n)}\\ +f(a_1)&\frac{(x-a_0)(x-a_2)(x-a_3)\ .\ .\ .\ (x-a_n)}{(a_1-a_0)(a_1-a_2)(a_1-a_3)\ .\ .\ .\ (a_1-a_n)}\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ +f(a_n)&\frac{(x-a_0)(x-a_1)\ .\ .\ .\ (x-a_{n-1})}{(a_n-a_0)(a_n-a_1)\ .\ .\ .\ (a_n-a_{n-1})},\end{aligned}$$

which is the formula required.

There is an infinite number of functions of x, each of which has the values $f(a_0)$, $f(a_1)$, . . ., $f(a_n)$ at a_0, a_1, . . ., a_n respectively. In the practical applications of mathematics, however, we consider only functions, such that if a_0, a_1, . . ., a_n are sufficiently close together, any one of the functions may be represented with tolerable accuracy by the polynomial P_n, for the range of values included between a_0, a_1, . . ., a_n. The formula may thus be used for interpolation.

19. The Remainder Term in Lagrange's Formula of Interpolation.*—Let $f(x)$ be a function of the real variable x defined in an interval to which belong the values x_0, x_1, . . ., x_n, and possessing in this interval the derivative of order n.

Consider the function $g(x)$, where

$$g(x) = \begin{vmatrix} f(x) & x^n & x^{n-1} & \dots & x & 1 \\ f(x_0) & x_0{}^n & x_0{}^{n-1} & \dots & x_0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ f(x_n) & x_n{}^n & x_n{}^{n-1} & \dots & x_n & 1 \end{vmatrix}$$

The determinant vanishes for the values x_0, x_1, . . ., x_n. By the differential calculus we see that since $g(x)$ vanishes for $(n+1)$ values of x, its derivative $g'(x)$ vanishes for n values of x. the second derivative for $(n-1)$ values, and so on; the nth derivative vanishing for one value of x in the interval. Thus there exists a value x intermediate between x_0, x_1, . . ., x_n such that $g^{(n)}(x) = 0$.

Forming the nth derivative of the determinant by differentiating the variable elements of the first row, we have:

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* Peano, *Scritti offerti ad E. D' Ovideo* (Turin, 1918), p. 333.

$$\begin{vmatrix} f^n(x) & n! & 0 & \cdot\ \cdot\ \cdot & 0 & 0 \\ f(x_0) & x_0^{\ n} & x_0^{\ n-1} & \cdot\ \cdot\ \cdot & x_0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f(x_n) & x_n^{\ n} & x_n^{\ n-1} & \cdot\ \cdot\ \cdot & x_n & 1 \end{vmatrix} = 0.$$

If we expand this determinant according to the elements of the first column and solve for $f(x_0)$ in the resulting equation, we find

$$f(x_0) = \frac{(x_0 - x_2)(x_0 - x_3) \ . \ . \ . \ (x_0 - x_n)}{(x_1 - x_2)(x_1 - x_3) \ . \ . \ . \ (x_1 - x_n)} f(x_1)$$

$$+ \frac{(x_0 - x_1)(x_0 - x_3) \ . \ . \ . \ (x_0 - x_n)}{(x_2 - x_1)(x_2 - x_3) \ . \ . \ . \ (x_2 - x_n)} f(x_2)$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$+ \frac{(x_0 - x_1)(x_0 - x_2) \ . \ . \ . \ (x_0 - x_{n-1})}{(x_n - x_1)(x_n - x_2) \ . \ . \ . \ (x_n - x_{n-1})} f(x_n)$$

$$+ \frac{(x_0 - x_1)(x_0 - x_2) \ . \ . \ . \ (x_0 - x_n)}{n!} f^{(n)}(x),$$

where x is some number intermediate between $x_0, x_1, \ . \ . \ ., x_n$. This is *Lagrange's formula with a remainder term.*

Examples on Chapter II

1. If $f(x) = \frac{1}{x^2}$, find the divided differences $f(a, b)$, $f(a, b, c)$, and $f(a, b, c, d)$.

2. If $f(x) = g(x) + h(x)$, where $g(x) = x^4$ and $h(x) = x^3$, verify that $f(5, 7, 11, 13) = g(5, 7, 11, 13) + h(5, 7, 11, 13)$.

3. Given the values

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

form the table of divided differences and extend it to include the values of the function for $x = 3$ and $x = 14$.

4. Find the function $f(x)$ in each of the following cases :

(*a*)

x	11	13	14	18	19	21
$f(x)$	1342	2210	2758	5850	6878	9282

(*b*)

x	16	17	19	23	29	31
$f(x)$	65536	83521	130321	279841	707281	923521

by means of a table of divided differences.

5. Calculate $f(1)$, given the values

x	0	2	3	6	7	9
$f(x)$	658503	704969	729000	804357	830584	884736

6. Assuming $f(x)$ to be a function of the fourth degree in x, find the value of $f(19)$ from the values

x	11	17	21	23	31
$f(x)$	14646	83526	194486	279846	923526

7. The values of a cubic function are 150, 392, 1452, 2366, and 5202, corresponding to the values of the argument 5, 7, 11, 13, 17 respectively. Apply the Lagrange formula to find the function when the argument has the values 9 and 6·5 respectively.

8. Find an expression for the function in each of the examples (6) and (7), using the Lagrange formula of interpolation.

CHAPTER III

CENTRAL-DIFFERENCE FORMULAE

20. **Central-Difference Notations.**—In this chapter we shall consider certain formulae of interpolation which employ differences taken nearly or exactly from a single horizontal line of the difference table. In order to express these simply it is convenient to modify the notation of the calculus of differences.

Several systems of modified notation are in use. One, which we shall frequently employ, was introduced by W. F. Sheppard* and will be understood from the following difference table. It is based on a symbol δ which may be regarded as equivalent to $\Delta E^{-\frac{1}{2}}$, where E as usual denotes the transition from any number to the number next below it in the difference table, *i.e.* $E = 1 + \Delta$.

Since $\delta \equiv \Delta E^{-\frac{1}{2}}$ and therefore $\Delta \equiv \delta E^{\frac{1}{2}}$, we may write $\Delta u_0 = \delta u_{\frac{1}{2}}$, $\Delta^2 u_0 = \delta^2 u_1$, $\Delta^3 u_0 = \delta^3 u_{\frac{3}{2}}, \ldots, \Delta^n u_0 = \delta^n u_{\frac{n}{2}}$, and so on. Rewriting the ordinary difference table, we obtain

Argument.	*Entry.*				
$a-2w$	u_{-2}				
		$\delta u_{-\frac{3}{2}}$			
$a-w$	u_{-1}		$\delta^2 u_{-1}$		
		$\delta u_{-\frac{1}{2}}$		$\delta^3 u_{-\frac{1}{2}}$	
a	u_0		$\delta^2 u_0$		$\delta^4 u_0$
		$\delta u_{\frac{1}{2}}$		$\delta^3 u_{\frac{1}{2}}$	
$a+w$	u_1		$\delta^2 u_1$		$\delta^4 u_1$
		$\delta u_{\frac{3}{2}}$		$\delta^3 u_{\frac{3}{2}}$	
$a+2w$	u_2		$\delta^2 u_2$		$\delta^4 u_2$

If we suppose each row of the difference table to be numbered with the suffix p of the corresponding entry u_p, or, in the case of a row situated midway between two entries u_p and u_{p+1}, to take the number $p+\frac{1}{2}$, we see that $\Delta^{2r} u_0$, the differences of even order of u_0, are represented in the central-difference notation by $\delta^{2r} u_r$, since they are situated

* *Proc. London Math. Soc.* **31** (1899), p. 459.

in the row r. The differences of odd order $\Delta^{2r+1}u_0$ are represented by the expression $\delta^{2r+1}u_{r+\frac{1}{2}}$ since they lie in the row $r+\frac{1}{2}$.

It is often required to find the arithmetic mean of two adjacent entries in the same column of differences. In the δ system of notation we indicate this mean by the symbol μ. Thus $\mu\delta u_0$ is defined to be $\frac{1}{2}(\delta u_{-\frac{1}{2}}+\delta u_{\frac{1}{2}})$, $\mu\delta^3 u_0$ is $\frac{1}{2}(\delta^3 u_{-\frac{1}{2}}+\delta^3 u_{\frac{1}{2}})$, and so on for the mean differences of the other entries. The mean differences may be inserted in the table to fill in the gaps that occur between the symbols of the quantities from which they are derived.

In another notation which was suggested by S. A. Joffe * the symbol $\triangle$ is used instead of δ. The notation is illustrated in the following difference table :

Argument.	*Entry.*				
$a-2w$	u_{-2}				
		$\triangle u_{-\frac{3}{2}}$			
$a-w$	u_{-1}		$\triangle^2 u_{-1}$		
		$\triangle u_{-\frac{1}{2}}$		$\triangle^3 u_{-\frac{1}{2}}$	
a	u_0		$\triangle^2 u_0$		$\triangle^4 u_0$
		$\triangle u_{\frac{1}{2}}$		$\triangle^3 u_{\frac{1}{2}}$	
$a+w$	u_1		$\triangle^2 u_1$		
		$\triangle u_{\frac{3}{2}}$			
$a+2w$	u_2				

21. **The Newton-Gauss Formula of Interpolation.**—Suppose that a function $f(a+xw)$ is given for the values

$$\ldots\ a-w,\ a,\ a+w,\ a+2w,\ \ldots$$

of its argument.

If in the Newton formula for unequal intervals we take $a_0=a$, $a_1=a+w$, $a_2=a-w$, $a_3=a+2w$, $a_4=a-2w$, and so on, and denote $a+xw$ by u, we obtain

$$\begin{aligned} f(u) = {} & f(a)+(u-a)f(a,a+w)+(u-a)(u-a-w)f(a,a+w,a-w) \\ & +(u-a)(u-a-w)(u-a+w)f(a,a+w,a-w,a+2w) \\ & +(u-a)(u-a-w)(u-a+w)(u-a-2w) \\ & \qquad f(a,\ a+w,\ a-w,\ a+2w,\ a-2w) \\ & +(u-a)(u-a-w)(u-a+w)(u-a-2w)(u-a+2w) \\ & \qquad f(a,\ a+w,\ a-w,\ a+2w,\ a-2w,\ a+3w). \\ & +\ldots \end{aligned} \tag{1}$$

* *Trans. Act. Soc. Amer.* **18** (1917), p. 91.

The divided differences contained in this equation may be written in the ordinary notation of differences as follows:

$$f(a, a+w) = \frac{1}{w}\Delta f(a),$$

$$f(a, a+w, a-w) = \frac{1}{2!\,w^2}\Delta^2 f(a-w),$$

$$f(a, a+w, a-w, a+2w) = \frac{1}{3!\,w^3}\Delta^3 f(a-w),$$

etc.

Equation (1) thus takes the form

$$\begin{aligned} f(a+xw) = f(a) + x\Delta f(a) &+ \frac{x(x-1)}{2!}\Delta^2 f(a-w) \\ &+ \frac{(x+1)x(x-1)}{3!}\Delta^3 f(a-w) \\ &+ \frac{(x+1)x(x-1)(x-2)}{4!}\Delta^4 f(a-2w) \\ &+ \frac{(x+2)(x+1)x(x-1)(x-2)}{5!}\Delta^5 f(a-2w) + \ldots \quad \text{(A)} \end{aligned}$$

This formula, which is one of the group of formulae known to Newton, is often called the *Gauss* formula.

The differences used in this formula are as nearly as possible in the horizontal line through $f(a)$ in the original difference table. The formula is therefore convenient for use when the value of the argument for which the function is required is near the middle of the tabulated values. This formula may be represented more simply by using the symbol $(n)_r$ to denote the binomial coefficient

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{r!},$$

so that it may be written

$$\begin{aligned} f(a+xw) = f(a) + x\Delta f(a) + (x)_2\Delta^2 f(a-w) + (x+1)_3\Delta^3 f(a-w) \\ + (x+1)_4\Delta^4 f(a-2w) + (x+2)_5\Delta^5 f(a-2w) + \ldots \quad \text{(B)} \end{aligned}$$

22. **The Newton-Gauss Backward Formula.**—From the formula of the last section another may be derived which is often used when x is measured in a negative direction from $f(a)$, *i.e.* towards decreasing values of the argument. Suppose we write $f(a-xw)$ in the form $f\{a+x(-w)\}$ and change the sign of w in the discussion of the last section. The

order of the arguments and corresponding entries is then reversed. Instead of $\Delta f(a)$ in the Newton-Gauss formula we now have $f(a-w)-f(a)$, or $-\Delta f(a-w)$; $\Delta^3 f(a-w)$ in the above formula becomes $-\Delta^3 f(a-2w)$; $\Delta^5 f(a-2w)$ becomes $-\Delta^5 f(a-3w)$, and so on. We thus obtain the formula

$$f(a-xw)=f(a)-x\Delta f(a-w)+(x)_2\Delta^2 f(a-w)-(x+1)_3\Delta^3 f(a-2w)$$
$$+(x+1)_4\Delta^4 f(a-2w)-(x+2)_5\Delta^5 f(a-3w)+\ldots$$

which has been called the *Newton-Gauss formula for negative interpolation*, or the *Newton-Gauss backward formula.*

23. **The Newton-Stirling Formula.**—In the Gauss formula

$$f(a+xw)=f(a)+x\Delta f(a)+\tfrac{1}{2}x(x-1)\Delta^2 f(a-w)$$
$$+\tfrac{1}{6}(x+1)x(x-1)\Delta^3 f(a-w)$$
$$+\tfrac{1}{24}(x+1)x(x-1)(x-2)\Delta^4 f(a-2w)+\ldots$$

the terms may be rearranged thus:

$$f(a+xw)=f(a)+x\{\Delta f(a)-\tfrac{1}{2}\Delta^2 f(a-w)\}+\frac{x^2}{2!}\Delta^2 f(a-w)$$
$$+\frac{x(x^2-1^2)}{3!}\{\Delta^3 f(a-w)-\tfrac{1}{2}\Delta^4 f(a-2w)\}$$
$$+\frac{x^2(x^2-1^2)}{4!}\Delta^4 f(a-2w)+\ldots$$

Suppose we replace the differences of even order within the brackets by differences of odd order, using the identities

$$\Delta^2 f(a-w)=\Delta f(a)-\Delta f(a-w),$$
$$\Delta^4 f(a-2w)=\Delta^3 f(a-w)-\Delta^3 f(a-2w),$$

and so on. We obtain the result

$$f(a+xw)=f(a)+x\frac{\Delta f(a)+\Delta f(a-w)}{2}+\frac{x^2}{2!}\Delta^2 f(a-w)$$
$$+\frac{x(x^2-1^2)}{3!}\;\frac{\Delta^3 f(a-w)+\Delta^3 f(a-2w)}{2}+\frac{x^2}{4!}(x^2-1^2)\Delta^4 f(a-2w)$$
$$+\frac{x(x^2-1^2)(x^2-2^2)}{5!}\;\frac{\Delta^5 f(a-2w)+\Delta^5 f(a-3w)}{2}$$
$$+\frac{x^2(x^2-1^2)(x^2-2^2)}{6!}\Delta^6 f(a-3w)+\ldots \qquad \text{(A)}$$

This formula, which was first given by Newton,* was afterwards studied by Stirling † and is called the *Newton-Stirling formula.*

The mean-differences $\frac{1}{2}\{\Delta f(a)+\Delta f(a-w)\}, \frac{1}{2}\{\Delta^3 f(a-2w)+\Delta^3 f(a-w)\}$, etc., are completely symmetrical with regard to increasing and decreasing arguments. This fact enables us to express the formula very concisely by means of the central-difference notation of § 20 :

$$u_x = u_0 + x\mu\delta u_0 + \frac{x^2}{2!}\delta^2 u_0 + \frac{x(x^2-1)}{3!}\mu\delta^3 u_0 + \frac{x^2(x^2-1)}{4!}\delta^4 u_0 + \frac{x(x^2-1)(x^2-2^2)}{5!}\mu\delta^5 u_0 + \dots \quad \text{(B)}$$

where $\mu\delta u_0 = \frac{1}{2}(\delta u_{-\frac{1}{2}} + \delta u_{\frac{1}{2}})$,
$\mu\delta^3 u_0 = \frac{1}{2}(\delta^3 u_{-\frac{1}{2}} + \delta^3 u_{\frac{1}{2}})$,
and so on.

24. **The Newton-Bessel Formula.**—In the Newton-Gauss formula

$$\begin{aligned} f(a+xw) = f(a) &+ x\Delta f(a) + \tfrac{1}{2}x(x-1)\Delta^2 f(a-w) \\ &+ \tfrac{1}{6}(x+1)x(x-1)\Delta^3 f(a-w) \\ &+ \tfrac{1}{24}(x+1)x(x-1)(x-2)\Delta^4 f(a-2w) + \dots; \end{aligned}$$

let us substitute for $\frac{1}{2}f(a)$, $\frac{1}{2}\Delta^2 f(a-w)$, $\frac{1}{2}\Delta^4 f(a-2w)$, etc., their values obtained from the identities

$$\begin{aligned} f(a) &= f(a+w) - \Delta f(a), \\ \Delta^2 f(a-w) &= \Delta^2 f(a) - \Delta^3 f(a-w), \\ \Delta^4 f(a-2w) &= \Delta^4 f(a-w) - \Delta^5 f(a-2w), \\ &\text{etc.} \end{aligned}$$

The above equation becomes

$$\begin{aligned} f(a+xw) &= \tfrac{1}{2}\{f(a)+f(a+w)\} + (x-\tfrac{1}{2})\Delta f(a) \\ &+ \frac{x(x-1)}{2!}\tfrac{1}{2}\{\Delta^2 f(a-w) + \Delta^2 f(a)\} + \frac{x(x-1)(x-\frac{1}{2})}{3!}\Delta^3 f(a-w) \\ &+ \frac{(x+1)x(x-1)(x-2)}{4!}\cdot\tfrac{1}{2}\{\Delta^4 f(a-2w) + \Delta^4 f(a-w)\} + \dots \quad \text{(A)} \end{aligned}$$

which is symmetrical with respect to the argument $(a+\frac{1}{2}w)$.

This formula, which was first given by Newton ‡ and later used by Bessel, is called the *Newton-Bessel formula.*

* Newton, *Methodus Differentialis* (1711), Prop. iii. Case i.

† Stirling, *Methodus Differentialis* (1730), Prop. xx.

‡ *Methodus Differentialis* (1711), Prop. iii. Case ii. ; Stirling, *Methodus Differentialis* (1730), Prop. xx. Case ii.

If in this formula we write $x - \frac{1}{2} = y$, it becomes

$$f(a + \tfrac{1}{2}w + yw) = \tfrac{1}{2}\{f(a) + f(a+w)\} + y\Delta f(a) + \frac{y^2 - \frac{1}{4}}{2!}\tfrac{1}{2}\{\Delta^2 f(a-w) + \Delta^2 f(a)\} + \frac{y(y^2-\frac{1}{4})}{3!}\Delta^3 f(a-w) + \frac{(y^2-\frac{1}{4})(y^2-\frac{9}{4})}{4!}\tfrac{1}{2}\{\Delta^4 f(a-2w) + \Delta^4 f(a-w)\} + \ldots \quad \text{(B)}$$

25. **Everett's Formula.**—When it is required to interpolate between $f(a)$ and $f(a+w)$ in the construction of tables by the subdivision of intervals, statisticians frequently use a formula due to Everett,* which is generally written in the form

$$u_x = \left[\xi + \frac{\xi(\xi^2-1)}{3!}\delta^2 + \frac{\xi(\xi^2-1)(\xi^2-4)}{5!}\delta^4 + \ldots\right]u_0 + \left[x + \frac{x(x^2-1)}{3!}\delta^2 + \frac{x(x^2-1)(x^2-4)}{5!}\delta^4 + \ldots\right]u_1,$$

where u_x denotes $f(a+xw)$, and ξ denotes $(1-x)$, and where as usual δ^2 denotes $\Delta^2 E^{-1}$. Thus for $u_{\frac{3}{4}}$, $x = \frac{3}{4}$, $\xi = 1 - \frac{3}{4} = \frac{1}{4}$.

This formula involves only even central differences of each of the two middle terms of the series between which the interpolation has to be made.

To prove this formula we eliminate from the Newton-Gauss formula

$$f(a+xw) = f(a) + x\Delta f(a) + (x)_2\Delta^2 f(a-w) + (x+1)_3\Delta^3 f(a-w) + (x+1)_4\Delta^4 f(a-2w) + (x+2)_5\Delta^5 f(a-2w) + \ldots$$

the differences of odd order by means of the relations

$$\Delta f(a) = f(a+w) - f(a), \quad \Delta^3 f(a-w) = \Delta^2 f(a) - \Delta^2 f(a-w),$$
$$\Delta^5 f(a-2w) = \Delta^4 f(a-w) - \Delta^4 f(a-2w) \ \ldots$$

The Newton-Gauss formula becomes

$$f(a+xw) = f(a) + x\{f(a+w) - f(a)\} + (x)_2\Delta^2 f(a-w) + (x+1)_3\{\Delta^2 f(a) - \Delta^2 f(a-w)\} + (x+1)_4\Delta^4 f(a-2w) + (x+2)_5\{\Delta^4 f(a-w) - \Delta^4 f(a-2w)\} + \ldots$$

Using the relation $(p+1)_{q+1} = (p)_{q+1} + (p)_q$, this equation may be written

* *Brit. Assoc. Rep.* (1900), p. 648 ; *J.I.A.* **35**, p. 452 (1901). Tables of the coefficients in this formula have been published in *Tracts for Computers*, No. V.

$$f(a+xw)=(1-x)f(a)+xf(a+w)+(x+1)_3\Delta^2f(a)-(x)_3\Delta^2f(a-w)$$
$$+(x+2)_5\Delta^4f(a-w)-(x+1)_5\Delta^4f(a-2w)+\ldots$$

Introducing central differences and rearranging the terms,

$$f(a+xw)=(1-x)f(a)-(x)_3\delta^2f(a)-(x+1)_5\delta^4f(a)-\ldots$$
$$+xf(a+w)+(x+1)_3\delta^2f(a+w)+(x+2)_5\delta^4f(a+w)+\ldots$$

If we now transform the coefficients of $f(a)$ by means of the relation $1-x=\xi$, so that $(x)_3=-(\xi+1)_3$, $(x+1)_5=-(\xi+2)_5$, etc., we have

$$f(a+xw)=\xi f(a)+(\xi+1)_3\delta^2f(a)+(\xi+2)_5\delta^4f(a)+\ldots$$
$$+xf(a+w)+(x+1)_3\delta^2f(a+w)+(x+2)_5\delta^4f(a+w)+\ldots$$

which is *Everett's formula for equal * intervals of the argument.*

26. **Example of Central-Difference Formulae.**—The following example illustrates the application of the various central-difference formulae:

To compute the value of $\log_{10}$ cosh 0·3655, *having given a table of values of $\log_{10}$ cosh x at intervals* 0·002 *of the argument.*

Forming the difference table, we see that the differences of the third order are approximately constant. The differences of the fourth order will, however, be taken into account since such a difference may affect the accuracy of the last figure of the result.

Argument.	*Entry.*				
0·360	0·0275 5462 3980				
		30061 3825			
0·362	278 5523 7805		152 8035		
		30214 1860		−2122	
0·364	281 5737 9665		152 5913		−13
		30366 7773		−2135	
0·366	284 6104 7438		152 3778		−3
		30519 1551		−2138	
0·368	287 6623 8989		152 1640		
		30671 3191			
0·370	290 7295 2180				

In *Everett's* formula put $x=\frac{3}{4}$, $\xi=\frac{1}{4}$, and $u_0=0{\cdot}0281\ 5737\ 9665$.

$$f(0{\cdot}3655)=\tfrac{1}{4}(281\ 5737\ 9665)+(-\tfrac{5}{128})(152\ 5913)+\tfrac{63}{8192}(-13)$$
$$+\tfrac{3}{4}(284\ 6104\ 7438)+(-\tfrac{7}{128})(152\ 3778)+\tfrac{77}{8192}(-3)$$
$$=283\ 8513\ 0494{\cdot}75-14\ 2937{\cdot}59-0{\cdot}13=283\ 8498\ 7557{\cdot}03.$$

$\therefore$ log cosh (0·3655) = 0·0283 8498 7557.

* Corresponding formulae for unequal intervals have been given by R. Todhunter, *J.I.A.* **50** (1916), p. 137, and by G. J. Lidstone, *Proc. Edin. Math. Soc.* **40** (1922), p. 26.

In the *Newton-Bessel* formula put $x=\frac{3}{4}$.

$$f(0{\cdot}3655)=\tfrac{1}{2}\begin{pmatrix}281\ 5737\ 9665\\+284\ 6104\ 7438\end{pmatrix}+\tfrac{1}{4}(30366\ 7773)$$

$$+(-\tfrac{3}{32})\tfrac{1}{2}\begin{pmatrix}152\ 5913\\+152\ 3778\end{pmatrix}-\tfrac{1}{128}(-2135)+\tfrac{35}{2048}\,\tfrac{1}{2}(-13-3)$$

$$=283\,0921\,3551{\cdot}5+7591\,6943{\cdot}25-142954{\cdot}27+16{\cdot}68-0{\cdot}14$$

$$=283\ 8498\ 7557{\cdot}02.$$

$\therefore$ log cosh $(0{\cdot}3655)=\underline{0{\cdot}0283\ 8498\ 7557}.$

By the *Newton-Gauss* formula

$$f(0{\cdot}3655)=281\ 5737\ 9665+\tfrac{3}{4}(3\ 0366\ 7773)+(-\tfrac{3}{32})\,(152\ 5913)$$

$$+(-\tfrac{7}{128})\,(-2135)+\tfrac{35}{2048}(-13)$$

$$=281\ 5737\ 9665+2277\ 50829{\cdot}75-14\ 3054{\cdot}34$$

$$+116{\cdot}76-0{\cdot}22$$

$$=283\ 8498\ 7556{\cdot}95.$$

$\therefore$ log cosh $(0{\cdot}3655)=\underline{0{\cdot}0283\ 8498\ 7557}.$

By the *Newton-Stirling* formula

$$f(0{\cdot}3655)=281\ \ 5737\ \ 9665+\tfrac{3}{4}\cdot\tfrac{1}{2}\begin{pmatrix}3\ 0214\ 1860\\+3\ 0366\ 7773\end{pmatrix}+\tfrac{9}{32}(152\ 5913)$$

$$+(-\tfrac{7}{128})\tfrac{1}{2}\begin{pmatrix}-2122\\-2135\end{pmatrix}+(-\tfrac{21}{2048})\,(-13)$$

$$=281\ 5737\ 9665+2\ 2717\ 8612{\cdot}38+42\ 9163{\cdot}03$$

$$+116{\cdot}40+{\cdot}13$$

$$=283\ 8498\ 7556{\cdot}94.$$

$\therefore$ log cosh $(0{\cdot}3655)=\underline{0{\cdot}0283\ 8498\ 7557}.$

27. **The Formulae of the preceding Sections may be expressed more concisely by means of the Central-Difference Notation of § 20.**

Everett's formula:

$$u_x=\xi u_0+(\xi+1)_3\delta^2u_0+(\xi+2)_5\delta^4u_0+\ldots+(\xi+r)_{2r+1}\delta^{2r}u_0+\ldots$$

$$+xu_1+(x+1)_3\delta^2u_1+(x+2)_5\delta^4u_1+\ldots+(x+r)_{2r+1}\delta^{2r}u_1+\ldots$$

The *Newton-Bessel* formula:

$$u_x=\mu u_{\frac{1}{2}}+(x-\tfrac{1}{2})\delta u_{\frac{1}{2}}+(x)_2\mu\delta^2u_{\frac{1}{2}}+\frac{x(x-1)\,(x-\frac{1}{2})}{3!}\delta^3u_{\frac{1}{2}}$$

$$+(x+1)_4\mu\delta^4u_{\frac{1}{2}}+\frac{(x+1)x(x-1)\,(x-2)\,(x-\frac{1}{2})}{5!}\delta^5u_{\frac{1}{2}}+\ldots$$

$$+\ldots+(x+r-1)_{2r}\mu\delta^{2r}u_{\frac{1}{2}}+(x+r-1)_{2r}\frac{x-\frac{1}{2}}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}+\ldots$$

The *Newton-Gauss* formula:

$$u_x=u_0+x\delta u_{\frac{1}{2}}+(x)_2\delta^2u_0+(x+1)_3\delta^3u_{\frac{1}{2}}+(x+1)_4\delta^4u_0+(x+2)_5\delta^5u_{\frac{1}{2}}$$

$$+\ldots+(x+r-1)_{2r}\delta^{2r}u_0+(x+r)_{2r+1}\delta^{2r+1}u_{\frac{1}{2}}+\ldots$$

The *Newton-Stirling* formula:

$$u_x = u_0 + x\mu\delta u_0 + \frac{x^2}{2!}\delta^2 u_0 + \frac{x(x^2-1)}{3!}\mu\delta^3 u_0 + \frac{x^2(x^2-1^2)}{4!}\delta^4 u_0$$
$$+ \frac{x(x^2-1^2)(x^2-2^2)}{5!}\mu\delta^5 u_0 + \frac{x^2(x^2-1^2)(x^2-2^2)}{6!}\delta^6 u_0 + \ldots$$
$$+ \tfrac{1}{2}\{(x+r)_{2r} + (x+r-1)_{2r}\}\delta^{2r} u_0 + (x+r)_{2r+1}\mu\delta^{2r+1} u_0 + \ldots$$

Newton-Gauss backward formula:

$$u_{-x} = u_0 - x\delta u_{-\frac{1}{2}} + (x)_2\delta^2 u_0 - (x+1)_3\delta^3 u_{-\frac{1}{2}}$$
$$+ (x+1)_4\delta^4 u_0 - (x+2)_5\delta^5 u_{-\frac{1}{2}} + \ldots$$
$$+ (x+r-1)_{2r}\delta^{2r} u_0 - (x+r)_{2r+1}\delta^{2r+1} u_{-\frac{1}{2}} + \ldots$$

28. **The Lozenge Diagram.**—We shall now give a method which enables us to find a large number of formulae of interpolation.

Let $(p)_q$ denote the quantity $\frac{p!}{q!(p-q)!}$, and let u_r denote the entry $f(a+rw)$. We obtain at once the relations

$$(p)_q = (p+1)_{q+1} - (p)_{q+1}, \quad (1)$$
$$\Delta^q u_{-r+1} - \Delta^q u_{-r} = \Delta^{q+1} u_{-r}, \quad (2)$$

and, combining these equations, we see that

$$(p)_q\{\Delta^q u_{-r+1} - \Delta^q u_{-r}\} = \{(p+1)_{q+1} - (p)_{q+1}\}\Delta^{q+1} u_{-r},$$

or

$$(p)_q\Delta^q u_{-r} + (p+1)_{q+1}\Delta^{q+1} u_{-r} = (p)_q\Delta^q u_{-r+1} + (p)_{q+1}\Delta^{q+1} u_{-r}. \quad (3)$$

Suppose we arrange these terms in the form of a "lozenge" so that the terms on the left-hand side of the equation lie along the two upper sides of the lozenge and the terms of the right-

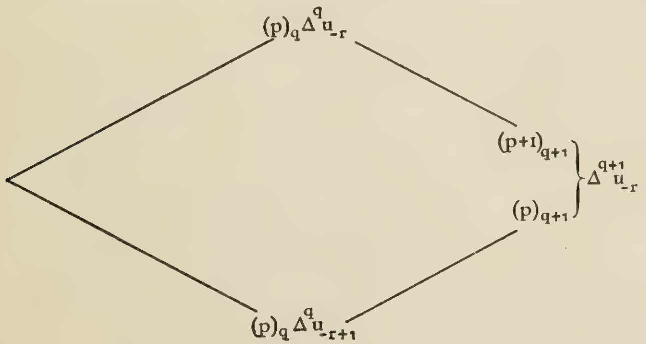

FIG. 2.

hand side along the lower sides. We obtain the above diagram in which a line directed from left to right joining two quantities denotes the *addition* of those quantities.

Equation (3) may be expressed by the statement that: *in travelling from the left-hand vertex to the right-hand vertex of the lozenge in the diagram, the sum of the elements which lie along the upper route is equal to the sum of the elements which lie along the lower route.*

It is evident that this statement may be extended. For example, let us place in contiguity the lozenges corresponding to

$$\begin{pmatrix} p=n \\ q=1 \\ r=1 \end{pmatrix} \begin{pmatrix} p=n-1 \\ q=1 \\ r=0 \end{pmatrix} \begin{pmatrix} p=n \\ q=2 \\ r=1 \end{pmatrix}$$

so that the upper vertices of the lozenges, which are of the form $(p)_q \Delta^q u_{-r}$, form a sort of difference table:

$$\begin{array}{ll} (n)_1 \Delta u_{-1} & \\ & (n)_2 \Delta^2 u_{-1} \\ (n-1)_1 \Delta u_0 & \end{array}$$

We obtain the following diagram:

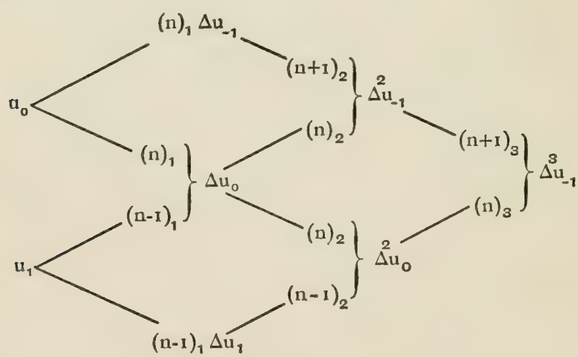

Fig. 3.

Applying the rule given by equation (3), it is evident that the sum of the elements along either of the following routes is the same:

$$\begin{aligned} & u_0 + (n)_1 \Delta u_{-1} + (n+1)_2 \Delta^2 u_{-1} + (n+1)_3 \Delta^3 u_{-1}, \\ & u_0 + (n)_1 \Delta u_0 + (n)_2 \Delta^2 u_{-1} + (n+1)_3 \Delta^3 u_{-1}, \\ & u_0 + (n)_1 \Delta u_0 + (n)_2 \Delta^2 u_0 + (n)_3 \Delta^3 u_{-1}. \end{aligned}$$

Since $u_0 + (n)_1\Delta u_0 = u_1 + (n-1)_1\Delta u_0$, we may form three other expressions beginning with the term u_1 instead of u_0 and equivalent to those already given, namely,

$$u_1 + (n-1)_1\Delta u_0 + (n)_2\Delta^2 u_{-1} + (n+1)_3\Delta^3 u_{-1}$$

and two similar expressions.

If we examine the structure of this diagram, it will be seen that the values of q and r in the expression $(p)_q\Delta^q u_{-r}$ are arranged in precisely the same way as for the differences $\Delta^q f(u - rw)$ in an ordinary difference table. The values of p are constant along any diagonal descending from left to right of the diagram, while along a diagonal ascending from left to right these values increase by unity at each vertex. The first value of p along either line radiating from u_0 is taken to be $p = n$.

By extending this diagram we arrive at the following, which

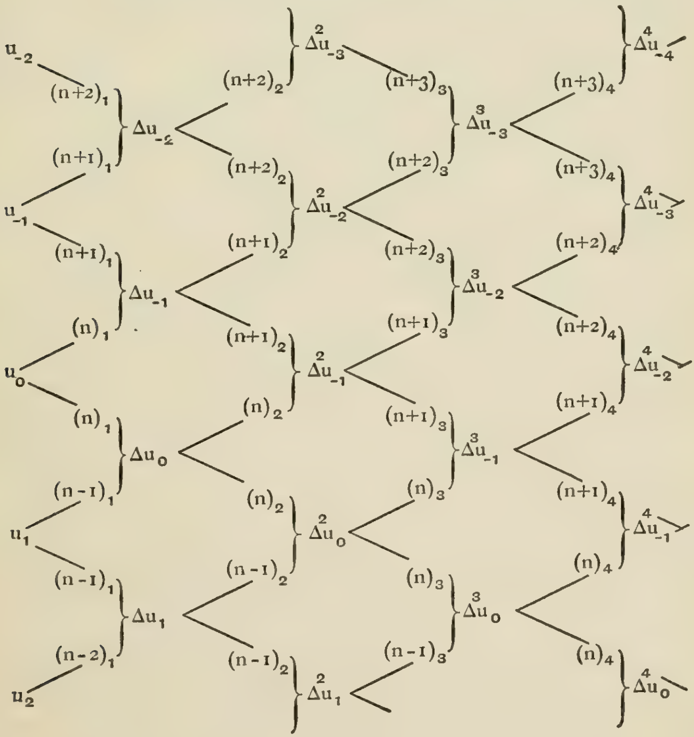

FIG. 4.

may be called a *lozenge* or "*Fraser*" diagram since it is a modification of one due to D. C. Fraser.*

Now the Gregory-Newton formula for u_n is the sum of the elements from u_0 along the downward sloping line to the line of zero differences. So $u_n =$ the sum of the elements from u_0 *along any route whatever* to the line of zero differences.

From the identity $u_0 + n\Delta u_0 = u_1 + (n-1)\Delta u_0$ it is evident that the value of u_n is unaltered if a route is selected starting from u_1 instead of from u_0. *In general the sum of the elements along any route proceeding from any entry u_r whatever to the line of zero differences is equal to u_n.*

Applying this rule, we have at once from the lozenge diagram

$$u_n = u_0 + (n)_1\Delta u_{-1} + (n+1)_2\Delta^2 u_{-2} + (n+2)_3\Delta^3 u_{-3} + (n+3)_4\Delta^4 u_{-4} + \dots \quad (4)$$

$$u_n = u_0 + (n)_1\Delta u_{-1} + (n+1)_2\Delta^2 u_{-1} + (n+1)_3\Delta^3 u_{-2} + (n+2)_4\Delta^4 u_{-2} + \dots \quad (5)$$

$$u_n = u_0 + (n)_1\Delta u_0 + (n)_2\Delta^2 u_{-1} + (n+1)_3\Delta^3 u_{-1} + (n+1)_4\Delta^4 u_{-2} + \dots \quad (6)$$

$$u_n = u_1 + (n-1)_1\Delta u_0 + (n)_2\Delta^2 u_0 + (n)_3\Delta^3 u_{-1} + (n+1)_4\Delta^4 u_{-1} + \dots \quad (7)$$

Rewriting equations (5), (6) in the central-difference notation, we find

$$u_n = u_0 + (n)_1\delta u_{-\frac{1}{2}} + (n+1)_2\delta^2 u_0 + (n+1)_3\delta^3 u_{-\frac{1}{2}} + (n+2)_4\delta^4 u_0 + \dots$$

and

$$u_n = u_0 + (n)_1\delta u_{\frac{1}{2}} + (n)_2\delta^2 u_0 + (n+1)_3\delta^3 u_{\frac{1}{2}} + (n+1)_4\delta^4 u_0 + \dots$$

which is the *Newton-Gauss* formula.

If we now take the mean of these values of u_n, we obtain the formula whose differences are along the row corresponding to u_0:

$$u_n = u_0 + (n)_1\tfrac{1}{2}(\delta u_{-\frac{1}{2}} + \delta u_{\frac{1}{2}}) + \tfrac{1}{2}\{(n+1)_2 + (n)_2\}\delta^2 u_0 + (n+1)_3\tfrac{1}{2}(\delta^3 u_{-\frac{1}{2}} + \delta^3 u_{\frac{1}{2}}) + \tfrac{1}{2}\{(n+2)_4 + (n+1)_4\}\delta^4 u_0 + \dots$$

or

$$u_n = u_0 + (n)_1\mu\delta u_0 + \tfrac{1}{2}n^2\delta^2 u_0 + \tfrac{1}{6}n(n^2-1)\mu\delta^3 u_0 + \tfrac{1}{24}n^2(n^2-1)\delta^4 u_0 + \dots$$

which is the *Newton-Stirling* formula.

The mean value of u_n from equations (6), (7) may be expressed either as Everett's formula or as the Newton-Bessel formula. Writing (6), (7) in the central-difference notation,

$$u_n = u_0 + (n)_1\delta u_{\frac{1}{2}} + (n)_2\delta^2 u_0 + (n+1)_3\delta^3 u_{\frac{1}{2}} + (n+1)_4\delta^4 u_0 + \dots + (n+r-1)_{2r}\delta^{2r} u_0 + (n+r)_{2r+1}\delta^{2r+1} u_{\frac{1}{2}} + \dots \quad (8)$$

$$u_n = u_1 + (n-1)_1\delta u_{\frac{1}{2}} + (n)_2\delta^2 u_1 + (n)_3\delta^3 u_{\frac{1}{2}} + (n+1)_4\delta^4 u_1 + \dots + (n+r-1)_{2r}\delta^{2r} u_1 + (n+r-1)_{2r+1}\delta^{2r+1} u_{\frac{1}{2}} + \dots \quad (9)$$

Taking the arithmetic mean of these values of u_n, we may eliminate

* *J.I.A.* **43** (1909), p. 238.

differences of *odd* order by applying the relations $(p)_q=(p+1)_{q+1}-(p)_{q+1}$ and $\delta^{2r+1}u_{\frac{1}{2}}=\delta^{2r}u_1-\delta^{2r}u_0$. The coefficient of $\delta^{2r}u_1$ takes the form $\frac{1}{2}\{(n+r-1)_{2r}+(n+r)_{2r+1}+(n+r-1)_{2r+1}\}$ or $(n+r)_{2r+1}$. The coefficient of $\delta^{2r}u_0$ becomes $\frac{1}{2}\{(n+r-1)_{2r}-(n+r)_{2r+1}-(n+r-1)_{2r+1}\}$ or $-(n+r-1)_{2r+1}$, and by substituting ξ for $(1-n)$ we see that

$$-(n+r-1)_{2r+1}=-(r-\xi)_{2r+1}=(\xi+r)_{2r+1}.$$

The arithmetic mean of equations (8), (9) may thus be written in the form

$$\begin{aligned}u_n=\xi u_0+(\xi+1)_3\delta^2u_0+(\xi+2)_5\delta^4u_0+\ldots+(\xi+r)_{2r+1}\delta^{2r}u_0+\ldots\\+nu_1+(n+1)_3\delta^2u_1+(n+2)_5\delta^4u_1+\ldots+(n+r)_{2r+1}\delta^{2r}u_1+\ldots\end{aligned}$$

which is *Everett's* formula.

Suppose, however, we find the arithmetic mean of the values of u_n in (8) and (9) and simplify the coefficients of differences of odd order in the resulting expression by means of the relation

$$\tfrac{1}{2}\{(n+r)_{2r+1}+(n+r-1)_{2r+1}\}=(n+r-1)_{2r}\cdot\frac{n-\frac{1}{2}}{2r+1}.$$

We now obtain the result

$$\begin{aligned}u_n=\mu u_{\frac{1}{2}}+(n-\tfrac{1}{2})\delta u_{\frac{1}{2}}+(n)_2\mu\delta^2u_{\frac{1}{2}}+\frac{n(n-1)(n-\frac{1}{2})}{3!}\delta^3u_{\frac{1}{2}}+(n+1)_4\mu\delta^4u_{\frac{1}{2}}+\ldots\\+(n+r-1)_{2r}\mu\delta^{2r}u_{\frac{1}{2}}+(n+r-1)_{2r}\frac{n-\frac{1}{2}}{2r+1}\delta^{2r+1}u_{\frac{1}{2}}+\ldots\end{aligned}$$

which is the *Newton-Bessel* formula.

29. Relative Accuracy of Central-Difference Formulae.—It is frequently necessary to use approximate formulae which terminate before the column of zero differences is reached. From the last section we have seen that the sums of the elements along any two routes which terminate at the same vertex are identical. If the routes terminate at two adjacent vertices $(p)_q\Delta^q u_{-r+1}$ and $(p)_q\Delta^q u_{-r}$ which are in the same "lozenge," the sums of the elements along these routes differ by $(p)_q(\Delta^q u_{-r+1}-\Delta^q u_{-r})$, *i.e.* by $(p)_q\Delta^{q+1}u_{-r}$. Extending this result to routes terminating in the same column of differences, for example, at Δ^4u_{-3} and Δ^4u_0, it is evident that the sums of the elements along these routes differ by $(n+2)_4\Delta^5u_{-3}+(n+1)_4\Delta^5u_{-2}+(n)_4\Delta^5u_{-1}$.

We shall now consider routes that lie along horizontal lines; these yield the formulae containing mean-differences. In the last section it was shown that a mean-difference formula is obtained by taking the arithmetic mean of the elements along two adjacent routes. From the mode of formation we see that the sums of the elements along such routes are identical as far as the vertices at the intersections of the routes. For example, the Newton-Gauss formula is equivalent to the Newton-Stirling formula as far as differences of *even* order, and it is also equivalent to the Newton-Bessel formula as far as differences of *odd* order. When a formula is curtailed, the question arises as to whether it is more advantageous to select a route which terminates at a mean difference or at an ordinary difference.

The following diagram represents the portion of the lozenge diagram along the row corresponding to u_0 and adjacent to the differences of order $2r$. Let A denote the mean difference $(n+r)_{2r+1}\mu\delta^{2r+1}u_0$, and let B denote the mean difference $(n+r-1)_{2r}\mu\delta^{2r}u_{\frac{1}{2}}$.

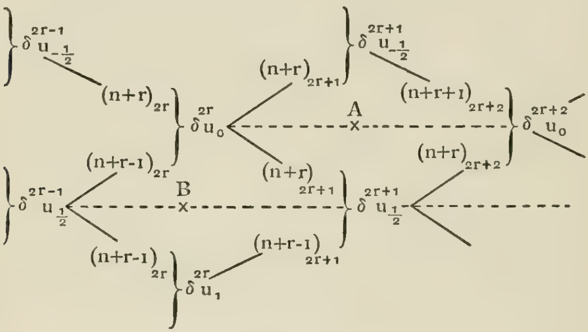

FIG. 5.

The route along the dotted line through A represents the *Newton-Stirling* formula and the route along the dotted line through B represents the *Newton-Bessel* formula. The *Newton-Gauss* formula, which is represented in the diagram by a zigzag intermediate route, is equivalent to the Stirling formula at the vertices $\delta^{2r}u_0$ and $\delta^{2r+2}u_0$, and it is also equivalent to the Bessel formula at the vertices $\delta^{2r-1}u_{\frac{1}{2}}$ and $\delta^{2r+1}u_{\frac{1}{2}}$.

Consider the three routes representing the Gauss and the Stirling formulae and the formula which contains the differences $\delta^{2r-1}u_{-\frac{1}{2}}$, $\delta^{2r}u_0$, $\delta^{2r+1}u_{-\frac{1}{2}}$, and $\delta^{2r+2}u_0$. If we suppose these formulae to be curtailed so that the last difference of each is of order $2r+1$, we may compare the accuracy of these formulae by ascertaining the magnitude of the neglected terms of order $(2r+2)$. The sum of the elements along either of the routes from the common vertex $\delta^{2r+2}u_0$ to the line of zero differences being the same, the most accurate formula is the one in which the neglected term of order $(2r+2)$ is the smallest. These terms are: $(n+r)_{2r+2}\delta^{2r+2}u_0$, $\frac{1}{2}\{(n+r)_{2r+2}+(n+r+1)_{2r+2}\}\delta^{2r+2}u_0$, $(n+r+1)_{2r+2}\delta^{2r+2}u_0$ respectively, and they are also arranged in ascending order of magnitude. The Newton-Gauss formula is therefore more accurate as far as mean differences of order $(2r+1)$, when further terms are neglected, than the corresponding Newton-Stirling formula passing through the same differences of even order; and both are more accurate than the formula containing the difference $\delta^{2r+1}u_{-\frac{1}{2}}$. In precisely the same way we see that the Bessel formula is more accurate than the Gauss formula as far as differences of even order when further terms are neglected. In general, *a central-difference formula terminating at a mean difference of the entry* u_p *is more accurate than a formula which is curtailed at the corresponding central-difference of* $u_{p-\frac{1}{2}}$, *and it is less accurate than a formula which is curtailed at the corresponding difference of* $u_{p+\frac{1}{2}}$.*

We shall now illustrate by an example the superiority which central-difference formulae generally have over other interpolation formulae.

Let it be required to find u_x, where $-\frac{1}{2}<x<\frac{1}{2}$. If we employ for this purpose an interpolation formula which proceeds according to central differences of u_0, and stop at the $(2r+1)$th term, the result is the same as if we employed Lagrange's formula with given values of $u_{-r}, u_{-r+1}, \ldots, u_r$, so that by § 19 the error is

$$\frac{(x+r)(x+r-1)\ldots(x-r)}{(2r+1)!}f^{(2r+1)}(\xi),$$

where ξ denotes some number between $a-rw$ and $a+rw$. If, on the other hand, we employ the Gregory-Newton formula, and stop at the $(2r+1)$th term, the result we thereby obtain is the same as if we employed Lagrange's formula with given values of $u_0, u_1, \ldots, u_{2r}$, so that the error is

$$\frac{x(x-1)\ldots(x-2r)}{(2r+1)!}f^{(2r+1)}(\eta),$$

where η denotes some number between a and $a+2rw$. Now $f^{(2r+1)}(\xi)$ does not, in most cases, differ greatly from $f^{(2r+1)}(\eta)$, but $(x+r)(x+r-1)\ldots(x-r)$ is much smaller than $x(x-1)\ldots(x-2r)$ in absolute value when $-\frac{1}{2}<x<\frac{1}{2}$. Thus the error is smaller in the former case than

* A detailed discussion of the accuracy of interpolation formulae is given in papers by W. F. Sheppard, *Proc. Lond. Math. Soc.* **4** (1906), p. 320, and **10** (1911), p. 139; D. C. Fraser, *J.I.A.* **50**, pp. 25-27; G. J. Lidstone, *Trans. Fac. Act.* **9** (1923).

in the latter. For this reason central-difference formulae are preferable to the ordinary formulae for advancing differences.

The following remarks* are of general application :

"Formulas which proceed to constant differences are exact, and are true for all values of n whether integral or fractional.

"Formulas which stop short of constant differences are approximations.

"Approximate formulas which terminate with the same difference are identically equal.

"Approximate formulas which terminate with distinct differences of the same order are not identical. The difference between them is expressed by the chain of lines necessary to complete the circuit."

30. Preliminary Transformations.

— In certain cases formulae of interpolation should not be used until some preliminary transformation has been effected. We shall illustrate this by two examples.

Ex. 1.—Suppose that it is required to find L sin 15″. We have from a table of logarithms the following entries :

θ.	*L sin* θ.			
0° 0′ 10″	5·6855749			
		3010300		
20″	5·9866049		−1249388	
		1760912		737864
30″	6·1626961		−511524	
		1249388		231236
40″	6·2876349		−280288	
		969100		
50″	6·3845449			

The differences are evidently very slowly convergent. One reason for this will be seen when it is remembered that when θ is small and $\theta'' = x$ radians, then $\sin x = x - \frac{1}{6}x^3 + \ldots$ and $x = \theta \sin 1''$ (nearly), so that $\text{L} \sin \theta = \text{L} \sin 1'' + \log \theta$ (nearly), and the differences of $\log \theta$ for the values 10, 20, 30, 40, 50 . . . of θ are very slowly convergent. We therefore calculate L sin θ when θ is small by adding the interpolated values of $\text{L}\left(\frac{\sin \theta}{\theta}\right)$, which has regular differences, and $\log \theta$, for which tables exist with smaller intervals of the argument.

Ex. 2.—Suppose it is required to interpolate between two terms of such a sequence as the following :

$$1, \quad \frac{r}{p}, \quad \frac{r(r+1)}{p(p+1)}, \quad \frac{r(r+1)(r+2)}{p(p+1)(p+2)}, \quad \frac{r(r+1)(r+2)(r+3)}{p(p+1)(p+2)(p+3)}, \ldots$$

where r and p are two widely different numbers.

* D. C. Fraser, *J.I.A.* **43** (1909), p. 238.

It is best to interpolate in the sequence of numerators

$$1,\ r,\ r(r+1),\ r(r+1)(r+2),\ .\ .\ .$$

and to interpolate separately in the sequence of denominators

$$1,\ p,\ p(p+1),\ p(p+1)(p+2)\ .\ .\ .$$

We then divide the former result by the latter, in order to obtain the required interpolated value.

Stirling (*Methodus Differentialis* (1730), Prop. xvii. Scholium) says: "As in common algebra the whole art of the analyst does not consist in the resolution of the equations but in bringing the problems thereto; so likewise in this analysis: there is less dexterity required in the performance of the process of interpolation than in the preliminary determination of the sequences which are best fitted for interpolation."

The general rule is to make such transformations as will make the interpolation as simple as possible.

Examples on Chapter III

1. Given

$$\begin{aligned} \sin 25^\circ\, 41'\, 40'' &= 0{\cdot}433\ 571\ 711\ 655\ 565 \\ \sin 25^\circ\, 42'\ \ 0'' &= 0{\cdot}433\ 659\ 084\ 587\ 544 \\ 20'' &= 0{\cdot}433\ 746\ 453\ 442\ 359 \\ 40'' &= 0{\cdot}433\ 833\ 818\ 219\ 189 \end{aligned}$$

find the value of sin 25° 42′ 10″ by the Newton-Gauss formula.

2. Find the value of log sin 0° 16′ 8″·5 having given

$$\begin{aligned} \log\sin 0^\circ\, 16'\, 7'' &= 7{\cdot}670\ 999\ 750\ 0 \\ 8'' &= 7{\cdot}671\ 448\ 629\ 9 \\ 9'' &= 7{\cdot}671\ 897\ 046\ 4 \\ 10'' &= 7{\cdot}672\ 345\ 000\ 2 \end{aligned}$$

using the Newton-Gauss formula.

Check your result by obtaining log sin 0° 16′ 8″·5 from the following data:

$$\begin{aligned} \log\sin 0^\circ\, 16'\, 6'' &= 7{\cdot}670\ 550\ 405\ 5 \\ 8'' &= 7{\cdot}671\ 448\ 629\ 9 \\ 10'' &= 7{\cdot}672\ 345\ 000\ 2 \\ 12'' &= 7{\cdot}673\ 239\ 524\ 3 \end{aligned}$$

3. Apply the Newton-Stirling formula to compute sin 25° 40′ 30″ from the table of values

$$\begin{aligned} \sin 25^\circ\, 40'\, 0'' &= 0{\cdot}433134785866963 \\ 20'' &= 0{\cdot}433222179172439 \\ 40'' &= 0{\cdot}433309568404859 \\ \sin 25^\circ\, 41'\, 0'' &= 0{\cdot}433396953563401 \\ 20'' &= 0{\cdot}433484334647243 \end{aligned}$$

and verify your answer, using the Newton-Bessel formula.

4. Given

$$\begin{aligned} \log 310 &= 2{\cdot}4913617 \\ 320 &= 2{\cdot}5051500 \\ 330 &= 2{\cdot}5185139 \\ 340 &= 2{\cdot}5314789 \\ 350 &= 2{\cdot}5440680 \\ 360 &= 2{\cdot}5563025 \end{aligned}$$

find the value of log 3375 by the Newton-Bessel formula, verifying the result by one or more other central-difference formulae and comparing it with the true value. [3·5282738.]

5. Show that the lozenge-diagram method really derives all the interpolation formulae by repeated summation by parts, *i.e.* by the use of the formulae

$$u_{x+1}\Delta v_x = \Delta(u_x v_x) - v_x \Delta u_x,$$

which is the analogue in the Calculus of Differences of the formula

$$\int u\,dv = uv - \int v\,du$$

in the Integral Calculus.

CHAPTER IV

APPLICATIONS OF DIFFERENCE FORMULAE

31. **Subtabulation.**—An important application of interpolation formulae is to the extension of tables of a function. Thus, supposing we already possess a table giving $\sin x$ at intervals of $1'$ of x, we might wish to construct a table giving $\sin x$ at intervals of $10''$ of x. This operation is called *subtabulation.* Subtabulation might evidently be performed by calculating each of the new values by ordinary interpolation, but when the new values are required in this wholesale fashion it is better to proceed otherwise, forming first the *differences* of the new sequence of values of the function, and then calculating the latter from those differences.*

Let $T_0, T_1, T_2, T_3, \ldots$ be a given sequence of entries in a table corresponding to intervals w of the argument, and let their successive differences be $\Delta T_0 = T_1 - T_0$, $\Delta^2 T_0 = T_2 - 2T_1 + T_0$, etc. Suppose it is desired to find the values of the function in question at intervals w/m of the argument so that $(m-1)$ intermediate values are to be interpolated between every two consecutive members of the set $T_0, T_1, T_2, \ldots$ Denote the sequences thus required by $t_0, t_1, t_2, \ldots$, so that $t_0 = T_0$, $t_m = T_1$, $t_{2m} = T_2$, $t_{3m} = T_3$, etc., and let the successive differences in the new sequence be

$$\Delta_1 t_0 = t_1 - t_0, \quad \Delta_1^2 t_0 = t_2 - 2t_1 + t_0, \text{ etc.,}$$

where Δ_1 is used instead of Δ to denote the operation of differencing in the new sequence. The differences in the new sequence may now be found in terms of the differences in the old sequence by the use of operators in the following way.

* Lagrange, *Œuvres*, 5, p. 663 (1792–3).

Denoting the initial value t_0 or T_0 by $f(a)$, we have by the Gregory-Newton interpolation formula:

$$t_1 = f(a + w/m) = T_0 + (1/m)_1\Delta T_0 + (1/m)_2\Delta^2 T_0 + (1/m)_3\Delta^3 T_0 + \ldots$$

and the operators Δ_1 and Δ are thus connected by the relation

$$\Delta_1 = (1/m)_1\Delta + (1/m)_2\Delta^2 + (1/m)_3\Delta^3 + \ldots \qquad (1)$$

Suppose for simplicity that $\Delta^4 T_0$ is the last non-zero difference of the original sequence, so that $\Delta^5 T_0 = 0$, $\Delta^6 T_0 = 0$, etc. Equation (1) gives

$$\Delta_1{}^s = \{(1/m)_1\Delta + (1/m)_2\Delta^2 + (1/m)_3\Delta^3 + (1/m)_4\Delta^4\}^s. \qquad (2)$$

If we now substitute the values $s = 1, 2, 3, 4$ in the last equation, we are able to determine all the differences of the new sequence in terms of the differences of the old sequence:

$$\Delta_1 t_0 = \frac{1}{m}\Delta T_0 + \frac{1-m}{2m^2}\Delta^2 T_0 + \frac{(1-m)(1-2m)}{6m^3}\Delta^3 T_0 + \frac{(1-m)(1-2m)(1-3m)}{24m^4}\Delta^4 T_0, \qquad (3)$$

$$\Delta_1{}^2 t_0 = \frac{1}{m^2}\Delta^2 T_0 + \frac{1-m}{m^3}\Delta^3 T_0 + \frac{(1-m)(7-11m)}{12m^4}\Delta^4 T_0, \qquad (4)$$

$$\Delta_1{}^3 t_0 = \frac{1}{m^3}\Delta^3 T_0 + \frac{3(1-m)}{2m^4}\Delta^4 T_0, \qquad (5)$$

$$\Delta_1{}^4 t_0 = \frac{1}{m^4}\Delta^4 T_0. \qquad (6)$$

When the differences are thus calculated, the entries t_1, t_2, t_3 may be derived in the usual way by simple addition. The values of t_m, t_{2m}, t_{3m}, . . . formed in this way should agree with the tabulated values T_1, T_2, T_3, . . .

Ex.—The logs of the numbers 1500, 1510, 1520, 1530, 1540 *being given to nine places of decimals, to find the logs of the integers between* 1500 *and* 1510.

The difference table of the original values is as follows:

No.	*log.*	Δ.	Δ^2.	Δ^3.	Δ^4.
1500	176091259				
		2885688			
1510	178976947		−19047		
		2866641		249	
1520	181843588		−18798		−4
		2847843		245	
1530	184691431		−18553		
		2829290			
1540	187520721				

Here $m=10$.

$\therefore\ \Delta_1{}^4=\frac{1}{10^4}(-4)=-0{\cdot}0004$ in the ninth place, which is negligible,

$$\Delta_1{}^3=\frac{1}{10^3}249+\frac{3(1-10)}{2\cdot 10^4}(-4)=0{\cdot}2544=0{\cdot}25$$

which is approximately constant,

$$\Delta_1{}^2=\frac{1}{10^2}(-19047)+\frac{(1-10)}{10^3}249+\frac{(1-10)(7-110)}{12\cdot 10^4}(-4)=-192{\cdot}74,$$

$$\Delta_1=\frac{1}{10}2885688+\frac{(1-10)}{2\cdot 10^2}(-19047)+\frac{(1-10)(1-20)}{6\cdot 10^3}249+\frac{(1-10)(1-20)(1-30)}{24\cdot 10^4}(-4)$$

$$\begin{aligned}&=288568{\cdot}8\\ &\quad\ \ 857{\cdot}115\\ &\quad\ \ \ \ 7{\cdot}0965\\ &\quad\ \ \ \ 0{\cdot}08265\\ &=289433{\cdot}094\end{aligned}$$

No.	*logs.*	Δ_1.	$\Delta_1{}^2$.	$\Delta_1{}^3$.
1500	176091259·1			
		289433·1		
1501	176380692·2		−192·74	
		289240·4		0·25
1502	176669932·6		−192·49	
		289047·9		0·25
1503	176958980·5		−192·24	
		288855·6		0·25
1504	177247836·1		−191·99	
		288663·6		0·25
1505	177536499·7		−191·74	
		288471·9		0·25
1506	177824971·6		−191·49	
		288280·4		0·25
1507	178113252·0		−191·24	
		288089·2		0·25
1508	178401341·2		−190·99	
		287898·2		0·25
1509	178689239·4		−190·74	
		287707·4		
1510	178976946·8			

The required new table is :

No.	*log.*	*No.*	*log.*
1500	3·176091259	1506	3·177824972
1501	3·176380692	1507	3·178113252
1502	3·176669933	1508	3·178401341
1503	3·176958981	1509	3·178689239
1504	3·177247836	1510	3·178976947
1505	3·177536500		

and the final value of log 1510 agrees with the original value.

32. **An Alternative Derivation.**—It is frequently convenient when dealing with a function whose degree is known to insert values of the function, intermediate to those already tabulated, by the following method:

Suppose, for example, that a function $f(x)$ may be represented by a polynomial of the third degree, and that values of the function are tabulated at intervals $w=10$ of the argument. Let it be required to insert values at an interval $w=1$. Using the notation of the last section, we have (by the Gregory-Newton formula)

$$\begin{aligned}
T_0 &= t_0,\\
T_1 &= t_{10} = t_0 + 10\Delta_1 t_0 + 45\Delta_1^2 t_0 + 120\Delta_1^3 t_0,\\
T_2 &= t_{20} = t_0 + 20\Delta_1 t_0 + 190\Delta_1^2 t_0 + 1140\Delta_1^3 t_0,\\
T_3 &= t_{30} = t_0 + 30\Delta_1 t_0 + 435\Delta_1^2 t_0 + 4060\Delta_1^3 t_0.
\end{aligned}$$

Differencing these equations, we see that

$$\begin{aligned}
\Delta T_0 &= 10\Delta_1 t_0 + 45\Delta_1^2 t_0 + 120\Delta_1^3 t_0,\\
\Delta T_1 &= 10\Delta_1 t_0 + 145\Delta_1^2 t_0 + 1020\Delta_1^3 t_0,\\
\Delta T_2 &= 10\Delta_1 t_0 + 245\Delta_1^2 t_0 + 2920\Delta_1^3 t_0.
\end{aligned}$$

Similarly

$$\begin{aligned}
\Delta^2 T_0 &= 100\Delta_1^2 t_0 + 900\Delta_1^3 t_0,\\
\Delta^2 T_1 &= 100\Delta_1^2 t_0 + 1900\Delta_1^3 t_0,\\
\therefore \Delta^3 T_0 &= 1000\Delta_1^3 t_0.
\end{aligned}$$

The leading term and its differences for the subdivided intervals are seen to be

$$\begin{aligned}
\Delta_1^3 t_0 &= \cdot 001\Delta^3 T_0,\\
\Delta_1^2 t_0 &= \cdot 01\Delta^2 T_0 - \cdot 009\Delta^3 T_0,\\
\Delta_1 t_0 &= \cdot 1\Delta T_0 - \cdot 045\Delta^2 T_0 + \cdot 0235\Delta^3 T_0,^*
\end{aligned}$$

from which the values $t_1, t_2, t_3, \ldots$ are formed by addition.

Ex.—Having given a table of values of log x at intervals of the argument $w=5$, *to insert between log* 6250 *and log* 6255 *the intermediate values of the function at intervals* $w=1$.

	Entry.	Δ.	Δ^2.
Put	$T_0 = \log 6250 = 3\cdot 7958800$		
		3473	
	$T_1 = \log 6255 = 3\cdot 7962273$		-3
		3470	
	$T_2 = \log 6260 = 3\cdot 7965743$		-2
		3468	
	$T_3 = \log 6265 = 3\cdot 7969211$		

The differences of the second order are approximately constant, so we assume log x to be a polynomial of the second degree.

$$\begin{aligned}
T_0 &= t_0 = 3\cdot 7958800,\\
T_1 &= t_5 = t_0 + 5\Delta_1 t_0 + 10\Delta_1^2 t_0,\\
T_2 &= t_{10} = t_0 + 10\Delta_1 t_0 + 45\Delta_1^2 t_0.\\
\Delta T_0 &= 5\Delta_1 t_0 + 10\Delta_1^2 t_0 = 3473,\\
\Delta T_1 &= 5\Delta_1 t_0 + 35\Delta_1^2 t_0.\\
\Delta^2 T_0 &= 25\Delta_1^2 t_0 = -3.
\end{aligned}$$

* These are precisely the set of equations of § 31 when $\Delta^3 T_0$, the third differences of the tabulated function, are assumed to be constant.

From these equations we obtain the values

$$\Delta_1{}^2 t_0 = -0{\cdot}12, \quad \Delta_1 t_0 = 694{\cdot}84,$$

expressed in units of the seventh decimal place.

Forming the difference table for the subdivided intervals,

Entry.	Δ_1.	$\Delta_1{}^2$.
log 62 50 = 37958800·00		
	694·84	
log 62 51 = 37959494·84		−0·12
	694·72	
log 62 52 = 37960189·56		−0·12
	694·60	
log 62 53 = 37960884·16		−0·12
	694·48	
log 62 54 = 37961578·64		−0·12
	694·36	
log 62 55 = 37962273·00		

We may now insert these values of the function in the table of values, thus:

$$\log 6251 = 3{\cdot}7959495$$
$$\log 6252 = 3{\cdot}7960190, \text{ etc.}$$

We may obtain without difficulty formulae for subtabulation based on central-difference formulae, or on Everett's formula. These are frequently to be preferred to the subtabulation formulae based on the Gregory-Newton formula.

Owing to the rapid accumulation of error in the higher orders of differences, care must be taken to include additional places of digits in the computations, as in the above examples.

33. **Estimation of Population for Individual Ages when Populations are given in Age Groups.**—We shall now find the values of a statistical quantity, such as the population of a given district, for individual years, when the sums of its values for quinquennial periods are given.*

Let . . ., u_{-2}, u_{-1}, u_0, u_1, u_2, . . . be the values of the quantity for individual years, and let the quinquennial sums be . . ., W_1, W_0, W_{-1}, . . . , so that

$$W_1 = u_7 + u_6 + u_5 + u_4 + u_3,$$
$$W_0 = u_2 + u_1 + u_0 + u_{-1} + u_{-2},$$
$$W_{-1} = u_{-3} + u_{-4} + u_{-5} + u_{-6} + u_{-7}.$$

It is required to find the value u_0 in terms of the W's.

* G. King, *J.I.A.* **43**, p. 109 (1909). See also **50**, p. 32.

The Newton-Stirling formula may be written

$$u_n = u_0 + n\frac{\Delta u_{-1} + \Delta u_0}{2} + \frac{n^2}{2}\Delta^2 u_{-1} + \frac{n(n^2-1)}{6}\,\frac{\Delta^3 u_{-2} + \Delta^3 u_{-1}}{2}$$
$$+ \frac{n^2(n^2-1)}{24}\Delta^4 u_{-2} + \dots$$

If we denote $u_n + u_{-n}$ by y_n and neglect the differences of the fourth and higher orders, we may write

$$y_n = 2u_0 + n^2\Delta^2 u_{-1}.$$

Therefore $$W_0 = u_0 + y_1 + y_2$$
$$= 5u_0 + 5\Delta^2 u_{-1},$$

and $$W_1 + W_{-1} = y_3 + y_4 + y_5 + y_6 + y_7$$
$$= 10u_0 + 135\Delta^2 u_{-1}.$$

Eliminating $\Delta^2 u_{-1}$ from the two last equations, u_0 may be expressed in terms of the W's:

$$125u_0 = 27W_0 - (W_{-1} + W_1),$$

or, writing $\Delta^2 W_{-1}$ for $(W_{-1} - 2W_0 + W_1)$, we obtain the result

$$125u_0 = 25W_0 - \Delta^2 W_{-1},$$

or $$u_0 = 0{\cdot}2W_0 - 0{\cdot}008\Delta^2 W_{-1}. \qquad (1)$$

Ex.—To find the value of the quantity for the middle year of the second quinquennium, when the following are three consecutive quinquennial sums: 36556 : 39387 : 41921.

Denote the given quinquennial sums by W_{-1}, W_0, W_1 respectively, and form a difference table.

$W_{-1} = 36\,556$		
	2831	
$W_0 = 39\,387$		-297
	2534	
$W_1 = 41\,921$		

The required quantity u_0 is therefore, by (1),

$$u_0 = 0{\cdot}2 \times 39\,387 - {\cdot}008\,(-297)$$
$$= 7877{\cdot}4 + 2{\cdot}4$$
$$= 7879{\cdot}8,$$

so $$u_0 = 7880.$$

The above formula may be extended to include the fourth differences of the W's when we neglect the differences of the u's of the sixth and higher orders.* We have now

* When the groups are unequal, we can proceed in a similar way, using divided differences.

$$\begin{aligned}y_n &= u_n + u_{-n}\\ &= 2u_0 + n^2\Delta^2 u_{-1} + \tfrac{1}{12}n^2(n^2-1)\Delta^4 u_{-2},\\ W_0 &= u_0 + y_1 + y_2\\ &= 5u_0 + 5\Delta^2 u_{-1} + \Delta^4 u_{-2}, \qquad (2)\\ W_1 + W_{-1} &= 10u_0 + 135\Delta^2 u_{-1} + 377\Delta^4 u_{-2},\\ W_2 + W_{-2} &= 10u_0 + 510\Delta^2 u_{-1} + 4627\Delta^4 u_{-2}.\end{aligned}$$

Eliminating u_0 from the three last equations, we have

$$\Delta^2 W_{-1} = 125\Delta^2 u_{-1} + 375\Delta^4 u_{-2},$$

and
$$\Delta W_1 - \Delta W_{-2} = 375\Delta^2 u_{-1} + 4250\Delta^4 u_{-2},$$

and eliminating $\Delta_2 u_{-1}$ from these two equations we find that

$$\Delta^4 u_{-2} = 0{\cdot}00032\Delta^4 W_{-2}$$

and
$$\Delta^2 u_{-1} = 0{\cdot}008\Delta^2 W_{-1} - 0{\cdot}00096\Delta^4 W_{-2}.$$

If we now substitute these values in equation (2), we obtain the result

$$u_0 = 0{\cdot}2(W_0 - 5\Delta^2 u_{-1} - \Delta^4 u_{-2}),$$

or
$$u_0 = 0{\cdot}2W_0 - 0{\cdot}008\Delta^2 W_{-1} + 0{\cdot}000896\Delta^4 W_{-2}.$$

This value of u_0 was also given by G. King.*

The following demonstration of a more general formula is due to G. J. Lidstone.

Let
$$W_0 = \sum_{-r}^{r} u_s, \quad W_1 = \sum_{r+1}^{3r+1} u_s, \text{ etc.},$$

and let
$$y_x = \sum_{p}^{(2r+1)x-r-1} u_s,$$

where p is some number independent of x. From these definitions we have at once

$$\Delta y_x = W_x$$

and
$$u_0 = y_{\frac{1}{2}+\frac{1}{2(2r+1)}} - y_{\frac{1}{2}-\frac{1}{2(2r+1)}}.$$

In Bessel's formula,

$$y_{\frac{1}{2}+m} = \frac{y_0+y_1}{2} + m\Delta y_0 + \frac{m^2-\frac{1}{4}}{2\,!}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2} + \frac{m(m^2-\frac{1}{4})}{3\,!}\Delta^3 y_{-1} + \ldots$$

put
$$m = \frac{1}{2(2r+1)}.$$

Form the difference $y_{\frac{1}{2}+m} - y_{\frac{1}{2}-m}$ and in the result substitute W and its

* *J.I.A.* **43**, p. 114.

differences for Δy and its differences. We thus obtain the required formula.

The result is

$$u_0 = 2mW_0 + \frac{m(m^2-\frac{1}{4})}{3!/2}\Delta^2 W_{-1} + \frac{m(m^2-\frac{1}{4})(m^2-\frac{9}{4})}{5!/2}\Delta^4 W_{-2} + \dots$$

which, when $2r+1=5$, becomes

$$u_0 = 0\cdot 2W_0 - 0\cdot 008\Delta^2 W_{-1} + 0\cdot 000896\Delta^4 W_{-2} + \dots$$

as found above.

34. **Inverse Interpolation.**—We shall now consider the process which is the inverse of direct interpolation, namely, that of finding the value of the *argument* corresponding to a given value of the *function* intermediate between two tabulated values, when a difference table of the function is given. This is known as *inverse interpolation.*

Let $f(a+xw)$ denote a particular value of the function of which the differences are tabulated. We now wish to find the value of the argument x corresponding to $f(a+xw)$; for this purpose it is best, if $-\frac{1}{4}<x<\frac{1}{4}$, to use Stirling's formula *

$$\begin{aligned} f(a+xw) = f(a) &+ x\tfrac{1}{2}\{\Delta f(a) + \Delta f(a-w)\} + \tfrac{1}{2}x^2\Delta^2 f(a-w) \\ &+ \tfrac{1}{6}x(x^2-1^2)\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &+ \tfrac{1}{24}x^2(x^2-1^2)\Delta^4 f(a-2w) + \dots \qquad (1) \end{aligned}$$

Dividing throughout by $\frac{1}{2}\{\Delta f(a) + \Delta f(a-w)\}$, the coefficient of x, equation (1) may be written in the form

$$x = m - \tfrac{1}{2}x^2 D_1 - \tfrac{1}{6}x(x^2-1)D_2 - \tfrac{1}{24}x^2(x^2-1)D_3 - \dots \qquad (2)$$

where $m = \{f(a+xw) - f(a)\}/\frac{1}{2}\{\Delta f(a) + \Delta f(a-w)\}$,

$D_1 = \{\Delta^2 f(a-w)\}/\frac{1}{2}\{\Delta f(a) + \Delta f(a-w)\}$,

$D_2 = \{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\}/\{\Delta f(a) + \Delta f(a-w)\}$,

and so on. We have now to solve equation (2) by successive approximations.

1st approximation: $x = m.$

Substituting this value in equation (2) we obtain the 2nd approximation:

$$x = m - \tfrac{1}{2}m^2 D_1 - \tfrac{1}{6}m(m^2-1)D_2 - \tfrac{1}{24}m^2(m^2-1)D_3 - \dots$$

This value of x is now substituted in equation (2) to form the 3rd approximation for x, and so on for further approximations.

* If $\frac{1}{4}<x<\frac{3}{4}$, Bessel's formula should be used.

Instead of solving equation (2) by successive approximations we may arrange it in the form

$$m = x + \tfrac{1}{2}x^2 D_1 + \tfrac{1}{6}x(x^2-1)D_2 + \tfrac{1}{24}x^2(x^2-1)D_3 + \ldots$$

We have merely to reverse this series to obtain a formula from which x may be found by direct substitution, namely,

$$\begin{aligned} x = {} & m(1+\tfrac{1}{6}D_2+\ldots) + m^2(-\tfrac{1}{2}D_1 + \tfrac{1}{24}D_3 - \tfrac{1}{4}D_1D_2 - \ldots) \\ & + m^3(\tfrac{1}{2}{D_1}^2 - \tfrac{1}{6}D_2 - \ldots) \\ & + \ldots \end{aligned}$$

As an example of inverse interpolation, suppose we wish to find the positive root of the equation *

$$z^7 + 28z^4 - 480 = 0.$$

Writing $y = z^7 + 28z^4 - 480$, and finding by a rough graph that the root is a little over 1·9, we construct the following difference table :

z.	y.	Δ.	Δ^2.	Δ^3.
1·90	−25·7140261			
		11·0886094		
1·91	−14·6254167		0·2293434	
		11·3179528		0·0041112
1·92	− 3·3074639		0·2334546	
		11·5514074		0·0041775
1·93	8·2439435		0·2376321	
		11·7890395		
1·94	20·0329830			

Evidently the root lies between 1·92 and 1·93, and therefore if the root be $1{\cdot}92 + 0{\cdot}01x$, we have by Stirling's formula in equation (1) :

$$0 = -3{\cdot}3074639 + 11{\cdot}4346801x + 0{\cdot}1167273x^2 + 0{\cdot}0006907(x^3 - x),$$
$$0 = -3{\cdot}3074639 + 11{\cdot}4339894x + 0{\cdot}1167273x^2 + 0{\cdot}0006907x^3.$$

Dividing throughout by the coefficient of x,

$$x = 0{\cdot}28926595 - 0{\cdot}0102088x^2 - 0{\cdot}0000604x^3.$$

1st approximation : $x = 0{\cdot}28926595$,

2nd approximation : $x = 0{\cdot}28926595 - 0{\cdot}0102088 \times 0{\cdot}083675 - 0{\cdot}0000604 \times 0{\cdot}0242$
$= 0{\cdot}28841027$,

3rd approximation : $x = 0{\cdot}28926595 - 0{\cdot}0102088 \times 0{\cdot}0831805 - 0{\cdot}0000604 \times 0{\cdot}0240$
$= 0{\cdot}28841533$.

The required root is 1·9228841533, correctly to 10 decimal places.

* This equation was suggested by W. B. Davis (*Ed. Times*, 1867, p. 108) but solved otherwise by him.

35. **The Derivatives of a Function.**—From the Gregory-Newton formula

$$f(a+xw)=f(a)+x\Delta f(a)+\frac{x(x-1)}{2!}\Delta^2 f(a)+\frac{x(x-1)(x-2)}{3!}\Delta^3 f(a)+\ldots \quad (1)$$

we have at once

$$\frac{f(a+xw)-f(a)}{xw}=\frac{1}{w}\{\Delta f(a)+\frac{x-1}{2}\Delta^2 f(a)+\frac{(x-1)(x-2)}{3!}\Delta^3 f(a)+\ldots\}.$$

If x is taken very small so that $xw \to 0$, the left-hand side of the equation is of the form $\{f(a+h)-f(a)\}/h$. The limiting value of this expression when $h \to 0$ is the *derivative* of the function $f(x)$ for the value a of its argument. We thus obtain

$$f'(a)=\frac{1}{w}\{\Delta f(a)-\tfrac{1}{2}\Delta^2 f(a)+\tfrac{1}{3}\Delta^3 f(a)-\tfrac{1}{4}\Delta^4 f(a)+\ldots\}. \quad (2)$$

The successive derivatives of the function may be obtained by the use of the differential calculus in the following way. Differentiating (1), we obtain

$$wf'(a+xw)=\Delta f(a)+\frac{2x-1}{2!}\Delta^2 f(a)+\frac{3x^2-6x+2}{3!}\Delta^3 f(a)+\frac{4x^3-18x^2+22x-6}{4!}\Delta^4 f(a)+\ldots$$

Also

$$w^2 f''(a+xw)=\Delta^2 f(a)+(x-1)\Delta^3 f(a)+\frac{6x^2-18x+11}{12}\Delta^4 f(a)+\ldots$$

and so on for derivatives of higher order.

Putting $x=0$ in this set of equations, we obtain the results

$$wf'(a)=\Delta f(a)-\tfrac{1}{2}\Delta^2 f(a)+\tfrac{1}{3}\Delta^3 f(a)-\tfrac{1}{4}\Delta^4 f(a)+\tfrac{1}{5}\Delta^5 f(a)-\tfrac{1}{6}\Delta^6 f(a)+\ldots$$

$$w^2 f''(a)=\Delta^2 f(a)-\Delta^3 f(a)+\tfrac{11}{12}\Delta^4 f(a)-\tfrac{5}{6}\Delta^5 f(a)+\tfrac{137}{180}\Delta^6 f(a)-\ldots$$

$$w^3 f'''(a)=\Delta^3 f(a)-\tfrac{3}{2}\Delta^4 f(a)+\tfrac{7}{4}\Delta^5 f(a)-\tfrac{15}{8}\Delta^6 f(a)+\ldots$$

$$w^4 f^{\text{iv}}(a)=\Delta^4 f(a)-2\Delta^5 f(a)+\tfrac{17}{6}\Delta^6 f(a)-\ldots$$

$$w^5 f^{\text{v}}(a)=\Delta^5 f(a)-\tfrac{5}{2}\Delta^6 f(a)+\ldots$$

$$w^6 f^{\text{vi}}(a)=\Delta^6 f(a)-\ldots$$

Ex.—*To find the first and second derivatives of* $\log_e x$ *at* $x = 500$.

x.	$\log_e x$.	Δ.	Δ^2.	Δ^3.	Δ^4.
500	6·214608				
		19803			
510	6·234411		−385		
		19418		15	
520	6·253829		−370		−1
		19048		14	
530	6·272877		−356		−1
		18692		13	
540	6·291569		−343		
		18349			
550	6·309918				

Here $w = 10$ and

$$10f'(500) = 0{\cdot}019803 + \tfrac{1}{2}(0{\cdot}000385) + \tfrac{1}{3}(0{\cdot}000015)$$
$$= 0{\cdot}020001.$$

$$\text{Also } 100f''(500) = -0{\cdot}000385 - 0{\cdot}000015 - \tfrac{11}{12}(0{\cdot}000001)$$
$$= -0{\cdot}000401.$$

Neglecting the last figure, which is liable to error, we obtain the results

$$f'(500) = 0{\cdot}002000$$
$$f''(500) = -0{\cdot}0000040.$$

We may find the formula for the nth derivative of a function otherwise, by using symbolic operators and expanding the function $f(a+w)$ by Taylor's Theorem.

Thus $$f(a+w) = f(a) + wf'(a) + \frac{w^2}{2!}f''(a) + \ldots \qquad (1)$$

If we denote $\frac{d}{dx}$, the operator for differentiation, by D, equation (1) becomes

$$f(a+w) = (1 + wD + \frac{w^2D^2}{2!} + \frac{w^3D^3}{3!} + \ldots)f(a),$$

or $$(1+\Delta)f(a) = e^{wD}f(a),$$

and $$1 + \Delta \equiv e^{wD}. \qquad (2)$$

Taking logarithms of each side of this equation,

$$wD = \log_e(1+\Delta)$$
$$= \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \ldots$$

or $$wf'(a) = \Delta f(a) - \tfrac{1}{2}\Delta^2 f(a) + \tfrac{1}{3}\Delta^3 f(a) - \ldots \qquad (3)$$

Also $$w^2D^2 = \{\log(1+\Delta)\}^2$$
$$= (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \ldots)^2.$$

Therefore $$w^2 f''(a) = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \ldots)^2 f(a)$$
$$= \Delta^2 f(a) - \Delta^3 f(a) + \tfrac{11}{12}\Delta^4 f(a) + \ldots \qquad (4)$$

and in general

$$w^n f^{(n)}(a) = (\Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \tfrac{1}{4}\Delta^4 + \ldots)^n f(a). \qquad (5)$$

36. **The Derivatives of a Function expressed in Terms of Differences which are in the same Horizontal Line.**— By differentiating Stirling's formula,

$$\begin{aligned} f(a+xw) = f(a) &+ x\tfrac{1}{2}\{\Delta f(a) + \Delta f(a-w)\} + \tfrac{1}{2}x^2\Delta^2 f(a-w) \\ &+ \tfrac{1}{6}x(x^2-1^2)\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &+ \tfrac{1}{24}x^2(x^2-1^2)\Delta^4 f(a-2w) \\ &+ \tfrac{1}{120}x(x^2-1^2)(x^2-2^2)\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} \\ &+ \tfrac{1}{720}x^2(x^2-1^2)(x^2-2^2)\Delta^6 f(a-3w), \end{aligned}$$

the differential coefficients may be represented by a rapidly converging series in terms of the horizontal differences. Thus

$$\begin{aligned} wf'(a+xw) &= \tfrac{1}{2}\{\Delta f(a) + \Delta f(a-w)\} + x\Delta^2 f(a-2w) \\ &+ \tfrac{1}{6}(3x^2-1)\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &+ \tfrac{1}{24}(4x^3-2x)\Delta^4 f(a-2w) \\ &+ \tfrac{1}{120}(5x^4-15x^2+4)\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} + \ldots \end{aligned}$$

$$\begin{aligned} w^2 f''(a+xw) &= \Delta^2 f(a-2w) + x\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &+ \tfrac{1}{24}(12x^2-2)\Delta^4 f(a-2w) \\ &+ \tfrac{1}{120}(20x^3-30x)\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} + \ldots \end{aligned}$$

Putting $x=0$ in these equations, we have

$$\begin{aligned} wf'(a) = \tfrac{1}{2}\{\Delta f(a) + \Delta f(a-w)\} &- \tfrac{1}{6}\cdot\tfrac{1}{2}\{\Delta^3 f(a-w) + \Delta^3 f(a-2w)\} \\ &+ \tfrac{1}{30}\cdot\tfrac{1}{2}\{\Delta^5 f(a-2w) + \Delta^5 f(a-3w)\} + \ldots \qquad (1) \end{aligned}$$

$$w^2 f''(a) = \Delta^2 f(a-w) - \tfrac{1}{12}\Delta^4 f(a-2w) + \tfrac{1}{90}\Delta^6 f(a-3w) + \ldots \qquad (2)$$

These equations give the value of the derivatives in terms of differences which are symmetrical as regards the direction of increasing and decreasing arguments.

In order to extend these results to derivatives of higher order we shall write Stirling's formula in the central-difference notation of § 20 as far as differences of the eighth order.

$$\begin{aligned} f(a+xw) = u_0 + x\mu\delta u_0 &+ \tfrac{1}{2}x^2\delta^2 u_0 + \tfrac{1}{6}x(x^2-1)\mu\delta^3 u_0 + \tfrac{1}{24}x^2(x^2-1)\delta^4 u_0 \\ &+ \tfrac{1}{120}x(x^2-1)(x^2-4)\mu\delta^5 u_0 + \tfrac{1}{720}x^2(x^2-1)(x^2-4)\delta^6 u_0 \\ &+ \tfrac{1}{5040}x(x^2-1)(x^2-4)(x^2-9)\mu\delta^7 u_0 \\ &+ \tfrac{1}{40320}x^2(x^2-1)(x^2-4)(x^2-9)\delta^8 u_0. \end{aligned}$$

When the right-hand side is arranged according to ascending powers of x, we obtain

$$\begin{aligned}f(a+xw) = u_0 &+ x(\mu\delta u_0 - \tfrac{1}{6}\mu\delta^3 u_0 + \tfrac{1}{30}\mu\delta^5 u_0 - \tfrac{1}{140}\mu\delta^7 u_0)\\ &+ x^2(\tfrac{1}{2}\delta^2 u_0 - \tfrac{1}{24}\delta^4 u_0 + \tfrac{1}{180}\delta^6 u_0 - \tfrac{1}{1120}\delta^8 u_0)\\ &+ x^3(\tfrac{1}{6}\mu\delta^3 u_0 - \tfrac{1}{24}\mu\delta^5 u_0 + \tfrac{7}{720}\mu\delta^7 u_0)\\ &+ x^4(\tfrac{1}{24}\delta^4 u_0 - \tfrac{1}{144}\delta^6 u_0 + \tfrac{7}{5760}\delta^8 u_0)\\ &+ x^5(\tfrac{1}{120}\delta^5 u_0 - \tfrac{1}{360}\mu\delta^7 u_0) + x^6(\tfrac{1}{720}\delta^6 u_0 - \tfrac{1}{2880}\delta^8 u_0). \quad (3)\end{aligned}$$

If both sides of this equation are differentiated and we substitute the value $x = 0$, we obtain the value of $wf'(a)$ as in equation (1); and the higher derivatives of $f(a)$ are formed by differentiating $wf'(a+xw)$, $w^2f''(a+xw)$, and so on.

The successive derivatives of $f(a)$ correct to differences of the eighth order are given by the following equations:

$$\begin{aligned}wf'(a) &= \mu\delta u_0 - \tfrac{1}{6}\mu\delta^3 u_0 + \tfrac{1}{30}\mu\delta^5 u_0 - \tfrac{1}{140}\mu\delta^7 u_0,\\ w^2f''(a) &= \delta^2 u_0 - \tfrac{1}{12}\delta^4 u_0 + \tfrac{1}{90}\delta^6 u_0 - \tfrac{1}{560}\delta^8 u_0,\\ w^3f'''(a) &= \mu\delta^3 u_0 - \tfrac{1}{4}\mu\delta^5 u_0 + \tfrac{7}{120}\delta^7 u_0,\\ w^4f^{(\text{IV})}(a) &= \delta^4 u_0 - \tfrac{1}{6}\delta^6 u_0 + \tfrac{7}{240}\delta^8 u_0,\\ w^5f^{(\text{V})}(a) &= \mu\delta^5 u_0 - \tfrac{1}{3}\mu\delta^7 u_0,\\ w^6f^{(\text{VI})}(a) &= \delta^6 u_0 - \tfrac{1}{4}\delta^8 u_0.\end{aligned}$$

We see that $wf'(a)$ is equal to the coefficient of x in (3) and, in general, *$w^nf^{(n)}(a)$ is equal to the coefficient of x^n in the equation* (3) *multiplied by n!*. This result might have been obtained at once by comparing (3) with Taylor's expansion of $f(a+xw)$.

37. **To express the Derivatives of a Function in Terms of its Divided Differences.**—We shall first find the derivative of a function $f(x)$ for the particular value a_0 of the argument x in terms of its divided differences. As shown at equation (3), § 13, we may write

$$\begin{aligned}f(u,a_0) = f(a_0,a_1) &+ (u-a_1)f(a_0,a_1,a_2) + (u-a_1)(u-a_2)f(a_0,a_1,a_2,a_3)\\ &+ \ldots + (u-a_1)(u-a_2)\ \ldots\ (u-a_{n-1})f(a_0,\ a_1,\ \ldots,\ a_n),\end{aligned}$$

where the divided differences of order beyond the nth are supposed negligible. If we put $u = a_0$, we have

$$\begin{aligned}f(a_0,\ a_0) = f(a_0,\ a_1) &+ (a_0-a_1)f(a_0,\ a_1,\ a_2)\\ &+ (a_0-a_1)(a_0-a_2)f(a_0,\ a_1,\ a_2,\ a_3) + \ldots\\ &+ (a_0-a_1)(a_0-a_2)\ \ldots\ (a_0-a_{n-1})f(a_0,\ a_1,\ \ldots,\ a_n). \quad (1)\end{aligned}$$

But in § 16 we found that

$$f(u) = f(a_0) + (u - a_0)f(a_0, a_0) + (u - a_0)^2 f(a_0, a_0, a_0) + (u - a_0)^3 f(a_0, a_0, a_0, a_0) + \ldots$$

and by Taylor's expansion,

$$f(u) = f(a_0) + (u - a_0)f'(a_0) + (u - a_0)^2 \frac{f''(a_0)}{2!} + (u - a_0)^3 \frac{f'''(a_0)}{3!} + \ldots$$

so that $f'(a_0) = f(a_0, a_0)$, $\frac{1}{2}f''(a_0) = f(a_0, a_0, a_0)$, and in general

$$f^{(n)}(a_0)/n! = f(a_0, a_0, \ldots, a_0),$$

which gives the nth derivative in terms of the divided difference of the nth order with repeated arguments.

Equation (1) thus becomes

$$f'(a_0) = f(a_0\, a_1) + (a_0 - a_1)f(a_0, a_1, a_2) + (a_0 - a_1)(a_0 - a_2)f(a_0, a_1, a_2, a_3) + \ldots + (a_0 - a_1)(a_0 - a_2) \ldots (a_0 - a_{n-1})f(a_0, a_1, \ldots, a_n), \quad (2)$$

which gives $f'(a_0)$ in terms of its successive divided differences.

As a special case of this formula when $a_1 = a_0 + w$, $a_2 = a_0 + 2w$, etc.

$$f'(a_0) = \frac{1}{w}\Delta f(a_0) + (-w)\frac{1}{2w^2}\Delta^2 f(a_0) + (-w)(-2w)\frac{1}{3!w^2}\Delta^3 f(a_0) + \ldots$$

or

$$wf'(a_0) = \Delta f(a_0) - \tfrac{1}{2}\Delta^2 f(a_0) + \tfrac{1}{3}\Delta^3 f(a_0) - \ldots$$

which is the formula of § 35.

A more general expression for the derivatives of a function in terms of its divided differences may be obtained from Newton's formula:

$$f(x) = f(a_0) + (x - a_0)f(a_0, a_1) + (x - a_0)(x - a_1)f(a_0, a_1, a_2) + (x - a_0)(x - a_1)(x - a_2)f(a_0, a_1, a_2, a_3) + \ldots$$

Denoting the factor $(x - a_n)$ by α_n, this equation becomes

$$f(x) = f(a_0) + \alpha_0 f(a_0, a_1) + \alpha_0\alpha_1 f(a_0, a_1, a_2) + \alpha_0\alpha_1\alpha_2 f(a_0, a_1, a_2, a_3) + \ldots + \alpha_0\alpha_1\alpha_2 \ldots \alpha_{n-1} f(a_0, a_1, a_2, \ldots, a_n). \quad (3)$$

Differentiating both sides of this equation, we see that

$$f'(x) = f(a_0, a_1) + (\alpha_0 + \alpha_1)f(a_0, a_1, a_2) + (\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_1\alpha_2)f(a_0, a_1, a_2, a_3) + \ldots \quad (4)$$

$$f''(x)/2! = f(a_0, a_1, a_2) + (\alpha_0 + \alpha_1 + \alpha_2)f(a_0, a_1, a_2, a_3) + (\alpha_0\alpha_1 + \alpha_0\alpha_2 + \alpha_0\alpha_3 + \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)f(a_0, a_1, a_2, a_3, a_4) + \ldots$$

$$f'''(x)/3! = f(a_0, a_1, a_2, a_3) + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)f(a_0, a_1, a_2, a_3, a_4) + \ldots \quad (5)$$

$$f^{\text{iv}}(x)/4! = f(a_0, a_1, a_2, a_3, a_4) + (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)f(a_0, a_1, a_2, a_3, a_4, a_5) + \ldots \quad (6)$$

and so on.

In these equations the coefficient of the divided differences of order r is a symmetric function of the quantities $a_0, a_1, a_2, \ldots, a_{r-1}$. In equation (3) this coefficient is of r dimensions, and after each differentiation its dimensions decrease by unity; so we see, therefore, that the coefficient of $f(a_0, a_1, \ldots, a_r)$ in the equation for $f^{(r)}(x)/r!$ is unity (*i.e.* zero dimension in $a_0, a_1, \ldots, a_{r-1}$), and all differences of lower order vanish.

If we suppose $a_0 = a_1 = a_2 = a_3 = \ldots = a_n$, we obtain the values given above: $f'(a_0) = f(a_0, a_0)$, $f''(a_0) = 2f(a_0, a_0, a_0)$, and so on.

Substituting in equation (4) the value $x = a_0$, we obtain equation (2), namely,

$$f'(a_0) = f(a_0, a_1) + (a_0 - a_1)f(a_0, a_1, a_2) + (a_0 - a_1)(a_0 - a_2)f(a_0, a_1, a_2, a_3) + \ldots$$

The latter equation is used when the derivative of a single value of the function is required; but when the derivatives of several values of the function are to be computed, we use equation (4).

Ex.—From the following table of values compute the third and fourth derivatives of $f(\theta)$ when the argument θ has the values 5, 14, *and* 23 *respectively.*

θ	2	4	9	13	16	21	29
$f(\theta)$	57	1345	66340	402052	1118209	4287844	21242820

We first form a table of divided differences:

θ.	$f(\theta)$.					
$a_0 = 2$	57					
		644				
$a_1 = 4$	1345		1765			
		12999		556		
$a_2 = 9$	66340		7881		45	
		83928		1186		1
$a_3 = 13$	402052		22113		64	
		238719		2274		1
$a_4 = 16$	1118209		49401		81	
		633927		4054		
$a_5 = 21$	4287844		114265			
		2119372				
$a_6 = 29$	21242820					

The function is evidently a polynomial of the 5th degree.
Tabulating the values of $a_0, a_1, a_2, \ldots$, we find

	$\theta = 5$.	$\theta = 14$.	$\theta = 23$.
a_0	3	12	21
a_1	1	10	19
a_2	− 4	5	14
a_3	− 8	1	10
a_4	−11	−2	7

From equation (5) we have at once

$$\tfrac{1}{6}f'''(\theta)=556+(a_0+a_1+a_2+a_3)45$$
$$+(a_0a_1+a_0a_2+a_0a_3+a_0a_4+a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)1,$$

so $$f'''(5)=1\ 626,\quad f'''(14)=12\ 102,\quad f'''(23)=32\ 298.$$

From equation (6) we have

$$\tfrac{1}{24}f^{\text{IV}}(\theta)=45+(a_0+a_1+a_2+a_3+a_4)1,$$

so $$f^{\text{IV}}(5)=624,\quad f^{\text{IV}}(14)=1704,\quad f^{\text{IV}}(23)=2784.$$

Examples on Chapter IV

1. The logs of the numbers 400, 410, 420, 430, 440 being given to seven places of decimals, find the logs of the integers between 400 and 410.

$$\begin{aligned}\log 400 &= 2{\cdot}6020600\\ \log 410 &= 2{\cdot}6127839\\ \log 420 &= 2{\cdot}6232493\\ \log 430 &= 2{\cdot}6334685\\ \log 440 &= 2{\cdot}6434527\end{aligned}$$

2. If $\Delta^r T_0$ is the last non-zero difference of the original sequence, so that $\Delta^{r+1}T_0=0$, $\Delta^{r+2}T_0=0$, . . ., show that the formulae for sub-tabulation are :

$$\Delta_1{}^r t_0=\frac{1}{m^r}\Delta^r T_0,^*$$

$$\Delta_1{}^{r-1}t_0=\frac{1}{m^{r-1}}\Delta^{r-1}T_0+\frac{(r-1)(1-m)}{2m^r}\Delta^r T_0,$$

$$\Delta_1{}^{r-2}t_0=\frac{1}{m^{r-2}}\Delta^{r-2}T_0+\frac{(r-2)(1-m)}{2m^{r-1}}\Delta^{r-1}T_0$$
$$+\left\{\frac{(r-2)(1-m)(1-2m)}{2.3.m^r}+\frac{(r-2)(r-3)(1-m)^2}{8m^r}\right\}\Delta^r T_0.$$

The differences of order higher than the rth in the new sequence are, of course, all zero.

3. The following are three consecutive quinquennial sums :

44133, 41921 and 39387.

* Mouton, an astronomer of Lyons, in 1670 noticed that if in a sequence whose rth differences are constant, say $=c$, intermediate terms are inserted corresponding to a division of each interval of the argument into m equal parts, then the new sequence has its rth difference constant and equal to c/m^r.

Find the value of the quantity for the middle year of the second quinquennium.

4. The populations for four consecutive age groups are given by the table of values

Age Group.	*Population.*
25 to 29 years (inclusive)	458572
30 to 34 years („)	441424
35 to 39 years („)	423123
40 to 44 years („)	402918

Estimate the populations of ages between 32 and 33 years, and between 37 and 38 years respectively.

5. Show that if

$$W_0 = u_{0/t} + u_{1/t} + \ldots + u_{(t-1)/t},$$

and in general

$$W_x = u_{\frac{x+0}{t}} + u_{\frac{x+1}{t}} + \ldots + u_{\frac{x+(t-1)}{t}},$$

then the individual value $u_{x/t}$ may be found from the groups of t individual values W_0, W_1, W_2, . . . and their differences by the formula

$$u_{x/t} = \frac{W_0}{t} + (2x - t + 1)\frac{\Delta W_0}{t^2.2!} + \{3x^2 + 3x(1-2t) + (1 - 3t + 2t^2)\}\frac{\Delta^2 W_0}{t^3.3!},$$

where third differences are neglected.*

6. In the following set of data h is the height above sea-level, p the barometric pressure. Calculate by a difference table the height at which $p = 29$ and the pressure when $h = 5280$.

$h =$	0	2753	4763	6942	10593
$p =$	30	27	25	23	20

7. Form a difference table from the following steam data, where p is pressure in lbs. per square inch.

θ° C	93·0	96·2	100·0	104·2	108·7
p	11·38	12·80	14·70	17·07	19·91

Calculate p when $\theta = 99°·1$ and determine by inverse interpolation the temperature at which $p = 15$.

8. Calculate the real root of the equation

$$x^3 + x - 3 = 0$$

by inverse interpolation.

* C. H. Forsyth, *Quarterly Publications of the American Statistical Association*, December 1916.

9. Find the differential coefficient of $\log_e x$ at $x=300$, given the table of values

x.	$\log_e x$.
300	5·703782474656
301	5·707110264749
302	5·710427017375
303	5·713732805509
304	5·717027701406
305	5·720311776607
306	5·723585101952
307	5·726847747587

Find from the above table the differential coefficient of $\log_e x$ at $x=302$.

10. Given the values

x.	y.
0	858·313740 095
1	869·645772 308
2	880·975826 766
3	892·303904 583
4	903·630006 875

find the value of $\dfrac{d^2y}{dx^2}$ when $x=0$.

11. Find $\dfrac{d^2y}{dz^2}$ when $z=1$, given the following values:

z.	y.
1	0·198669
2	0·295520
3	0·389418
4	0·479425
5	0·564642
6	0·644217

12. Apply the central-difference formulae of § 36 to compute the first and second derivatives of $\log_e 304$, having given the table of values of Ex. 9.

13. From the following data compute the first four derivatives of the function y corresponding to the argument $x=11$:

x.	y.
2	108 243 219
5	121 550 628
9	141 158 164
13	163 047 364
15	174 900 628
21	214 358 884

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(*The numerals refer to pages.*)

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